Decay and Continuity of Boltzmann Equation in Bounded Domains

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Abstract

Boundaries occur naturally in kinetic equations and boundary effects are crucial for dynamics of dilute gases governed by the Boltzmann equation. We develop a mathematical theory to study the time decay and continuity of Boltzmann solutions for four basic types of boundary conditions: inflow, bounce-back reflection, specular reflection, and diffuse reflection. We establish exponential decay in L^∞ norm for hard potentials for general classes of smooth domains near an absolute Maxwellian. Moreover, in convex domains, we also establish continuity for these Boltzmann solutions away from the grazing set of the velocity at the boundary. Our contribution is based on a new L^2 decay theory and its interplay with delicate L^∞ decay analysis for the linearized Boltzmann equation, in the presence of many repeated interactions with the boundary.

1   Introduction

Boundary effect plays a crucial role in the dynamics of gases governed by the Boltzmann equation:

$$\partial_t F + v \cdot \nabla_x F = Q(F, F) \tag{1}$$

where $F(t, x, v)$ is the distribution function for the gas particles at time $t \geq 0$, position $x \in \Omega$, and $v \in \mathbf{R}^3$. Throughout this paper, the collision operator takes the form

$$
\begin{aligned}
Q(F_1, F_2) &= \int_{\mathbf{R}^3} \int_{\mathbf{S}^2} |v - u|^\gamma F_1(u') F_2(v') q_0(\theta) d\omega du \\
&\quad - \int_{\mathbf{R}^3} \int_{\mathbf{S}^2} |v - u|^\gamma F_1(u) F_2(v) q_0(\theta) d\omega du \\
&\equiv Q_{\text{gain}}(F_1, F_2) - Q_{\text{loss}}(F_1, F_2),
\end{aligned}
\tag{2}
$$

where $u' = u + (v - u) \cdot \omega$, $v' = v - (v - u) \cdot \omega$, $\cos \theta = (u - v) \cdot \omega / |u - v|$, $0 \leq \gamma \leq 1$ (hard potential) and $0 \leq q_0(\theta) \leq C |\cos \theta|$ (angular cutoff). The mathematical study of the particle-boundary interaction in a bounded domain and its effect

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on the global dynamics is one of the fundamental problems in the Boltzmann theory. There are four basic types of boundary conditions for $F(t, x, v)$ at the boundary $\partial \Omega$: (1) In-flow injection: in which the incoming particles are prescribed; (2) bounce-back reflection: in which the particles bounce back at the reverse the velocity; (3) specular reflection: in which the particle bounce back specularly; (4) diffuse reflection (stochastic): in which the incoming particles are a probability average of the outgoing particles. Due to its importance, there have been many contributions in the mathematical study of different aspects of the Boltzmann boundary value problems [A], [AC], [AEMN], [AEP], [AH], [C2], [C3], [CC], [De], [Gui], [H], [LY], [MS], [M], [US], [YZ], among others, see also references in the books [C1], [CIP] and [U1].

According to Grad (p243, [Gr1]), one of the basic problems in the Boltzmann study is to prove existence and uniqueness of its solutions, as well as their time-decay toward an absolute Maxwellian, in the presence of compatible physical boundary conditions in a general domain. In spite of those contributions to the study of Boltzmann boundary problems, there are few mathematical results of uniqueness, regularity, and time decay-rate for Boltzmann solutions toward an Maxwellian. In [SA], it was announced that Boltzmann solutions near a Maxwellian would decay exponentially to it in a smooth bounded convex domain with specular reflection boundary conditions. Unfortunately, we are not aware of any complete proof for such a result over the last thirty years [U2]. Recently, important progress has been made in [DeV] and [V] to establish almost exponential decay rate for Boltzmann solutions with large amplitude for general collision kernels and general boundary conditions, provided certain a-priori strong Sobolev estimates can be verified. Even though these estimates had been established for spatially periodic domains [G3-4] near Maxwellians, their validity is completely open for the Boltzmann solutions, even local in time, in a bounded domain. As a matter of fact, such kind of strong Sobolev estimates may not be expected for a general non-convex domain [G4]. This is because even for simplest kinetic equations with the differential operator $v \cdot \nabla_x$, the phase boundary $\partial \Omega \times \mathbf{R}^3$ is always characteristic but not uniformly characteristic at the grazing set $\gamma_0 = \{(x, v) : x \in \partial \Omega, \text{ and } v \cdot n(x) = 0\}$ where $n(x)$ being the outward normal at x. Hence it is very challenging and delicate to obtain regularity from general theory of hyperbolic PDE. Moreover, in comparison with the half-space problems studied, for instance in [LY], [YZ], the complication of the geometry makes it difficult to employ spatial Fourier transforms in x. There are many cycles (bouncing characteristics) interacting with the boundary repeatedly, and analysis of such cycles is one of the key mathematical difficulties.

The purpose of this article is to develop an unified $L^2 - L^\infty$ theory in the near Maxwellian regime, to establish exponential decay toward a normalized Maxwellian $\mu = e^{-\frac{|v|^2}{2}}$, for all four basic types of the boundary conditions and rather general domains. Consequently, uniqueness among these solutions can be obtained. For convex domains, these solutions are shown to be continuous away from the singular grazing set γ_0.

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1.1 Domain and Characteristics

Throughout this paper, we define $\Omega = \{x : \xi(x) < 0\}$ is connected and bounded with $\xi(x)$ being a smooth function. We assume $\nabla \xi(x) \neq 0$ at the boundary $\xi(x) = 0$. The outward normal vector at $\partial\Omega$ is given by

$$n(x) = \frac{\nabla \xi(x)}{|\nabla \xi(x)|}, \tag{3}$$

and it can be extended smoothly near $\partial\Omega = \{x : \xi(x) = 0\}$. We say Ω is real analytic if ξ is real analytic in x. We define Ω is strictly convex if there exists $c_\xi > 0$ such that

$$\partial_{ij}\xi(x)\zeta^i\zeta^j \geq c_\xi |\zeta|^2 \tag{4}$$

for all x such that $\xi(x) \leq 0$, and all $\zeta \in \mathbf{R}^3$. We say that Ω has a rotational symmetry, if there are vectors x_0 and ϖ, such that for all $x \in \partial\Omega$

$$\{(x - x_0) \times \varpi\} \cdot n(x) \equiv 0. \tag{5}$$

We denote the phase boundary in the space $\Omega \times \mathbf{R}^3$ as $\gamma = \partial\Omega \times \mathbf{R}^3$, and split it into outgoing boundary γ_+, the incoming boundary γ_-, and the singular boundary γ_0 for grazing velocities:

$$
\begin{aligned}
\gamma_+ &= \{(x, v) \in \partial\Omega \times \mathbf{R}^3 : \quad n(x) \cdot v > 0\}, \\
\gamma_- &= \{(x, v) \in \partial\Omega \times \mathbf{R}^3 : \quad n(x) \cdot v < 0\}, \\
\gamma_0 &= \{(x, v) \in \partial\Omega \times \mathbf{R}^3 : \quad n(x) \cdot v = 0\}.
\end{aligned}
$$

Given (t, x, v), let $[X(s), V(s)] = [X(s; t, x, v), V(s; t, x, v)] = [x + (s - t)v, v]$ be the trajectory (or the characteristics) for the Boltzmann equation (1):

$$\frac{dX(s)}{ds} = V(s), \qquad\qquad \frac{dV(s)}{ds} = 0. \tag{6}$$

with the initial condition: $[X(t; t, x, v), V(t; t, x, v)] = [x, v]$. For any (x, v) such that $x \in \bar{\Omega}, v \neq 0$, we define its **backward exit time** $t_{\mathbf{b}}(x, v) \geq 0$ to be the the last moment at which the back-time straight line $[X(s; 0, x, v), V(s; 0, x, v)]$ remains in $\bar{\Omega}$:

$$t_{\mathbf{b}}(x, v) = \sup\{\tau \geq 0 : x - \tau v \in \bar{\Omega}\}. \tag{7}$$

We therefore have $x - t_{\mathbf{b}}v \in \partial\Omega$ and $\xi(x - t_{\mathbf{b}}v) = 0$. We also define

$$x_{\mathbf{b}}(x, v) = x(t_{\mathbf{b}}) = x - t_{\mathbf{b}}v \in \partial\Omega. \tag{8}$$

We always have $v \cdot n(x_{\mathbf{b}}) \leq 0$.

1.2 Boundary Condition and Conservation Laws

In terms of the standard perturbation f such that $F = \mu + \sqrt{\mu} f$, the Boltzmann equation can be rewritten as

$$\{\partial_t + v \cdot \nabla + L\} f = \Gamma(f, f), \qquad f(0, x, v) = f_0(x, v),$$

where the standard linear Boltzmann operator see [G] is given by

$$Lf \equiv \nu f - Kf = -\frac{1}{\sqrt{\mu}} \{Q(\mu, \sqrt{\mu} f) + Q(\sqrt{\mu} f, \mu)\} \tag{9}$$

with the collision frequency $\nu(v) \equiv \int |v - u|^\gamma \mu(u) q_0(\theta) du d\theta \backsim \{1 + |v|\}^\gamma$ for $0 \le \gamma \le 1$; and

$$\Gamma(f_1, f_2) = \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu} f_1, \sqrt{\mu} f_2) \equiv \Gamma_{\text{gain}}(f_1, f_2) - \Gamma_{\text{loss}}(f_1, f_2). \tag{10}$$

In terms of f, we formulate the boundary conditions as

(1) The in-flow boundary condition: for $(x, v) \in \gamma_-$

$$f|_{\gamma_-} = g(t, x, v) \tag{11}$$

(2) The bounce-back boundary condition: for $x \in \partial\Omega$,

$$f(t, x, v)|_{\gamma_-} = f(t, x, -v) \tag{12}$$

(3) Specular reflection: for $x \in \partial\Omega$, let

$$R(x)v = v - 2(n(x) \cdot v)n(x), \tag{13}$$

and

$$f(t, x, v)|_{\gamma_-} = f(x, v, v - 2(n(x) \cdot v)n(x)) = f(x, v, R(x)v) \tag{14}$$

(4) Diffusive reflection: assume the natural normalization,

$$c_\mu \int_{v \cdot n(x) > 0} \mu(v) |n(x) \cdot v| dv = 1, \tag{15}$$

then for $(x, v) \in \gamma_-$,

$$f(t, x, v)|_{\gamma_-} = c_\mu \sqrt{\mu(v)} \int_{v' \cdot n(x) > 0} f(t, x, v') \sqrt{\mu(v')} \{n_x \cdot v'\} dv'. \tag{16}$$

For both the bounce-back and specular reflection conditions (12) and (14), it is well-known that both mass and energy are conserved for (1). Without loss of generality, we may always assume that the mass-energy conservation laws hold for $t \ge 0$, in terms of the perturbation f:

$$\int_{\Omega \times \mathbf{R}^3} f(t, x, v) \sqrt{\mu} dx dv = 0, \tag{17}$$

$$\int_{\Omega \times \mathbf{R}^3} |v|^2 f(t, x, v) \sqrt{\mu} dx dv = 0. \tag{18}$$

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Moreover, if the domain Ω has *any* axis of rotation symmetry (5), then we further assume the corresponding conservation of angular momentum is valid for all $t \geq 0$:

$$\int_{\Omega \times \mathbf{R}^3} \{(x - x_0) \times \varpi\} \cdot v f(t, x, v) \sqrt{\mu} dx dv = 0. \tag{19}$$

For the diffuse reflection (16), the mass conservation (17) is assumed to be valid.

1.3   Main Results

We introduce the weight function

$$w(v) = (1 + \rho|v|^2)^\beta e^{\theta|v|^2}. \tag{20}$$

where $0 \leq \theta < \frac{1}{4}, \rho > 0$ and $\beta \in \mathbf{R}^1$.

Theorem 1 *Assume $w^{-2}\{1 + |v|\}^3 \in L^1$ in (20). There exists $\delta > 0$ such that if $F_0 = \mu + \sqrt{\mu}f_0 \geq 0$, and*

$$||wf_0||_\infty + \sup_{0 \leq t \leq \infty} e^{\lambda_0 t}||wg(t)||_\infty \leq \delta,$$

with $\lambda_0 > 0$, then there there exists a unique solution $F(t, x, v) = \mu + \sqrt{\mu}f \geq 0$ to the inflow boundary value problem (11) for the Boltzmann equation (1). There exists $0 < \lambda < \lambda_0$ such that

$$\sup_{0 \leq t \leq \infty} e^{\lambda t}||wf(t)||_\infty \leq C\{||wf_0||_\infty + \sup_{0 \leq t \leq \infty} e^{\lambda_0 t}||wg(s)||_\infty\}.$$

Moreover, if Ω is strictly convex (4), and if $f_0(x, v)$ is continuous except on γ_0, and $g(t, x, v)$ is continuous in $[0, \infty) \times \{\partial\Omega \times \mathbf{R}^3 \setminus \gamma_0\}$ satisfying

$$f_0(x, v) = g(x, v) \quad on \ \gamma_-,$$

then $f(t, x, v)$ is continuous in $[0, \infty) \times \{\bar{\Omega} \times \mathbf{R}^3 \setminus \gamma_0\}$.

Theorem 2 *Assume $w^{-2}\{1 + |v|\}^3 \in L^1$ in (20). Assume the conservation of mass (17) and energy (18) are valid for f_0. Then there exists $\delta > 0$ such that if $F_0(x, v) = \mu + \sqrt{\mu}f_0(x, v) \geq 0$ and $||wf_0||_\infty \leq \delta$, there exists a unique solution $F(t, x, v) = \mu + \sqrt{\mu}f(t, x, v) \geq 0$ to the bounce-back boundary value problem (12) for the Boltzmann equation (1) such that*

$$\sup_{0 \leq t \leq \infty} e^{\lambda t}||wf(t)||_\infty \leq C||wf_0||_\infty.$$

Moreover, if Ω is strictly convex (4), and if initially $f_0(x, v)$ is continuous except on γ_0 and

$$f_0(x, v) = f_0(x, -v) \ on \ \partial\Omega \times \mathbf{R}^3 \setminus \gamma_0,$$

then $f(t, x, v)$ is continuous in $[0, \infty) \times \{\bar{\Omega} \times \mathbf{R}^3 \setminus \gamma_0\}$.

Theorem 3 *Assume $w^{-2}\{1+|v|\}^3 \in L^1$ in (20). Assume that ξ is both strictly convex (4) and analytic, and the mass (17) and energy (18) are conserved for f_0. In the case of Ω has any rotational symmetry (5), we require that the corresponding angular momentum (19) is conserved for f_0. Then there exists $\delta > 0$ such that if $F_0(x,v) = \mu + \sqrt{\mu}f_0(x,v) \geq 0$ and $||wf_0||_\infty \leq \delta$, there exists a unique solution $F(t,x,v) = \mu + \sqrt{\mu}f(t,x,v) \geq 0$ to the specular boundary value problem (14) for the Boltzmann equation (1) such that*

$$\sup_{0 \leq t \leq \infty} e^{\lambda t} ||wf(t)||_\infty \leq C||wf_0||_\infty.$$

Moreover, if $f_0(x,v)$ is continuous except on γ_0 and

$$f_0(x,v) = f_0(x, R(x)v) \text{ on } \partial\Omega$$

then $f(t,x,v)$ is continuous in $[0,\infty) \times \{\bar{\Omega} \times \mathbf{R}^3 \setminus \gamma_0\}$.

Theorem 4 *Assume (15). There is $\theta_0(\nu_0) > 0$ such that*

$$\theta_0(\nu_0) < \theta < \frac{1}{4} , \quad \text{and} \quad \rho \quad \text{is sufficiently small} \quad (21)$$

for weight function w in (20). Assume the mass conservation (17) is valid for f_0. If $F_0(x,v) = \mu + \sqrt{\mu}f_0(x,v) \geq 0$ and $||wf_0||_\infty \leq \delta$ sufficiently small, then there exists a unique solution $F(t,x,v) = \mu + \sqrt{\mu}f(t,x,v) \geq 0$ to the diffuse boundary value problem (16) for the Boltzmann equation (1) such that

$$\sup_{0 \leq t \leq \infty} e^{\lambda t} ||wf(t)||_\infty \leq C||wf_0||_\infty.$$

Moreover, if ξ is strictly convex, and if $f_0(x,v)$ is continuous except on γ_0 with

$$f_0(x,v)|_{\gamma_-} = c_\mu \sqrt{\mu} \int_{\{n_x \cdot v' > 0\}} f_0(x,v')\sqrt{\mu(v')}\{n(x) \cdot v'\}dv'$$

then $f(t,x,v)$ is continuous in $[0,\infty) \times \{\bar{\Omega} \times \mathbf{R}^3 \setminus \gamma_0\}$.

1.4   Velocity Lemma and Analyticity

In section 2, we first establish some important analytical tools. The first Velocity Lemma 5 plays the most important role in the study of continuity and the cycles (bouncing generalized trajectories) in the specular case. It implies that in a strictly convex domain (4), the singular set γ_0 can not be reached via the trajectory $\frac{dx}{dt} = v$, $\frac{dv}{dt} = 0$ from interior points inside Ω, and hence γ_0 does not really participate or interfere with the interior dynamics. By Lemma 6, no singularity would be created from γ_0 and it is possible to perform calculus for the back-time exit time $t_{\mathbf{b}}(x,v)$. This is the foundation for future regularity study. Moreover, the Velocity Lemma 5 also provides the lower bound away from the singular set γ_0, which leads to the estimates for repeating bounces in

the specular reflection cases. Such a Velocity Lemma 5 was first discovered in [G3-4], in the study of regularity of Vlasov-Poisson (Maxwell) system with flat geometry. It then was generalized in [HH] for Vlasov-Poisson system in a ball, and it is the starting point for the recent final resolution to the Vlasov-Poisson in a general convex domain [HV] with specular boundary condition.

Lemma 7 gives refined estimates for the operator K_w with $w \backsim \mu^{-1/2}$. Similar estimates were established in [SG]. Lemma 8 states that the zero set of a analytic function is of measure zero unless such a analytic function is identically zero. This provides a very convenient tool to verify certain geometric conditions of general domains for particularly specular reflections.

1.5 L^2 Decay Theory

Since no spatial Fourier transform is available, we first establish linear L^2 exponential decay estimates in Section 3 via a functional analytical approach. It turns out that it suffices to establish the following finite-time estimate (Proposition 11)

$$\int_0^1 ||\mathbf{P}f(s)||_\nu^2 ds \leq M \left\{ \int_0^1 ||\{\mathbf{I}-\mathbf{P}\}f(s)||_\nu^2 + \text{boundary contributions} \right\} \quad (22)$$

for any solution f to the linear Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f + Lf = 0, \qquad f(0,x,v) = f_0(x,v) \quad (23)$$

with all four boundary conditions (11), (12), (14) and (16). Here for any fixed (t,x), the standard projection $\mathbf{P}$ onto the hydrodynamic part is given by

$$\begin{aligned}
\mathbf{P}f &= \{a(t,x) + b(t,x) \cdot v + c(t,x)|v|^2\}\sqrt{\mu(v)}, &(24)\\
\mathbf{P}_a f &= a(t,x)\sqrt{\mu(v)}, \quad \mathbf{P}_\mathbf{b}f = b(t,x)v\sqrt{\mu(v)}, \quad \mathbf{P}_c f = c(t,x)|v|^2\sqrt{\mu(v)},
\end{aligned}$$

and $||\cdot||_\nu$ is the weighted L^2 norm with the collision frequency $\nu(v)$.

Similar types of estimates like (22), but with strong Sobolev norms, have been established in recent years [G1] via so-called the macroscopic equations for the coefficients a, b and c. The key of the analysis was based on the ellipticity for b which satisfies the Poisson's equation $\Delta b = \partial^2\{\mathbf{I}-\mathbf{P}\}f$, where ∂^2 is some second order differential operators. In the presence of the boundary condition $b \cdot n(x) = 0$ (bounce-back and specular) or $b \equiv 0$ (inflow and diffuse) at $\partial\Omega$, such an ellipticity is very difficult to employ for the weak L^2, instead of H^1, estimate for b in (22). This is due to lack of regularity of b in (22), even the trace of b is hard to define. Instead, we employ the hyperbolic (transport) feature rather than elliptic feature of the problem to prove (22). By a method of contradiction, we can find f_k such that if (22) is not valid, then the normalized $Z_k(t,x,v) \equiv \frac{f_k(t,x,v)}{\sqrt{\int_0^1 ||\mathbf{P}f_k(s)||_\nu^2 ds}}$ satisfies $\int_0^1 ||\mathbf{P}Z_k(s)||_\nu^2 ds \equiv 1$, and

$$\int_0^1 ||(\mathbf{I}-\mathbf{P})Z_k(s)||_\nu^2 ds \leq \frac{1}{k}. \quad (25)$$

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Denote a weak limit of Z_k to be Z, we expect that $Z = \mathbf{P}Z = 0$, by each of the four boundary conditions. The key is to prove that $Z_k \to Z$ strongly to reach a contradiction. By the averaging Lemma [DL2], we know that $Z_k(s) \to Z$ strongly in the interior of Ω. As expected, the most delicate part is to exclude possible concentration near the boundary $\partial\Omega$. Since Z_k is a solution to the transport equation, it then follows (Proposition 17) that near $\partial\Omega$, on set of the non-grazing velocity $v \cdot n(x) \neq 0$ can be reached via a trajectory from the interior of Ω, which implies that Z_k can be controlled on such non-grazing set with no concentration. On the other hand, over the remaining almost grazing set $v \cdot n(x) \sim 0$, thanks to the fact (25), we know that

$$Z_k \sim \mathbf{P}Z_k = \{a_k(t,x) + b_k(t,x) \cdot v + c_k(t,x)|v|^2\}\sqrt{\mu(v)}.$$

We observe that such special form of velocity distribution $\mathbf{P}Z_k$ can *not* have concentration on the almost grazing set $v \cdot n(x) \sim 0$ (Lemma 15), and we therefore conclude (22). Clearly, the hyperbolic or the transport property is crucial to control boundary behaviors via the interior compactness of Z_k.

1.6 L^∞ Decay Theory

Section 4 is devoted to the study of linear L^∞ decay for all four different types of boundary conditions: in-flow, bounceback, specular and diffuse (stochastic) reflection. In order to control the nonlinear term $\Gamma(f, f)$, we need to estimate the weighted L^∞ of wf. We recall that $L = \nu - K$, and study the L^∞ (pointwise) decay of the linear Boltzmann equation (23). We denote a weight function

$$h(t, x, v) = w(v)f(t, x, v), \tag{26}$$

and study the equivalent linear Boltzmann equation:

$$\{\partial_t + v \cdot \nabla_x + \nu - K_w\}h = 0, \qquad h(0, x, v) = h_0(x, v) \equiv wf_0, \tag{27}$$

where

$$K_w h = wK(\frac{h}{w}), \tag{28}$$

together with various boundary conditions (11), (12), (14), or (16). In bounceback, specular, diffuse reflection as well as the inflow case with $g \equiv 0$, we denote the semigroup $U(t)h_0$ to be the solution to (27), and the semigroup $G(t)h_0$ to be the solution to the simpler transport equation without collision K_w :

$$\{\partial_t + v \cdot \nabla_x + \nu\}h = 0, \qquad h(0, x, v) = h_0(x, v) = wf_0. \tag{29}$$

Notice that neither $G(t)$ nor $U(t)$ is a strongly continuous semigroup in L^∞ [U1].

We first obtain explicit representation of $G(t)$ in the presence of various boundary conditions. We then can obtain the explicit exponential decay estimate for $G(t)$. Moreover, we also establish the continuity for $G(t)$ with a forcing

term q if Ω is strictly convex (4) based on the Velocity Lemma 5. To study the L^∞ decay for $U(t)$, we make use of the Duhamal Principle :

$$U(t) = G(t) + \int_0^t G(t - s_1) K_w U(s_1) ds_1. \qquad (30)$$

Following the pioneering work of Vidav [Vi], we iterate (30) back to get:

$$U(t) = G(t) + \int_0^t G(t-s_1) K_w G(s_1) ds_1 + \int_0^t \int_0^{s_1} G(t-s_1) K_w G(s_1-s) K_w U(s) ds ds_1. \qquad (31)$$

The idea is to estimate the last double integral in terms of the L^2 norm of $f = \frac{h}{w}$, which decays exponential by L^2 decay theory in Section 2. The key difficulty lies in the presence of different boundary conditions which could lead to complicated bouncing trajectories. Each of the boundary condition presents different difficulties.

Section 4.1 is devoted to the study of inflow boundary condition (11), in which the back-time trajectory is either from the initial plane or from the boundary. Even though when $g \neq 0$, the solution operators for (29) and (27) are not semigroups, for any (t, x, v), a similar representation as $G(t - s_1) K_w G(s_1 - s) K_w U(s)$ is still possible. With the compact property of K_w (Lemma 7), we are led to the main contribution in (31) roughly of the form

$$\int_0^t \int_0^{s_1} \int_{v',v''\text{bounded}} |h(s, X(s; s_1, X(s_1; t, x, v), v'), v'')| dv' dv'' ds ds_1. \qquad (32)$$

The v' integral is estimated by a change of variable introduced in [Vi], (see also in [LY])

$$y \equiv X(s; s_1, X(s_1; t, x, v), v') = x - (t - s_1)v - (s_1 - s)v'. \qquad (33)$$

Since $\det(\frac{dy}{dv'}) \neq 0$ almost always true, the v' and v''−integration in (32) can be bounded by $(h = wf)$

$$\int_{\Omega,v'' \text{ bounded}} |h(s, y, v'')| dy dv'' \leq C \left(\int_{\Omega,v'' \text{ bounded}} |f(s, y, v'')|^2 dy dv'' \right)^{1/2}.$$

For bounce-back, specular or diffuse reflections, the characteristic trajectories repeatedly interact with the boundary. Instead of $X(s; t, x, v)$, we should use the generalized characteristics, defined as cycles, $X_{\mathbf{cl}}(s; t, x, v)$ in (32) as in Definitions 21, 30 and 36. The key question is wether or not the change of variable

$$y \equiv X_{\mathbf{cl}}(s; s_1, X_{\mathbf{cl}}(s_1; t, x, v), v') \qquad (34)$$

is valid, i.e., to determine if it is almost always true

$$\det \left\{ \frac{dX_{\mathbf{cl}}(s; s_1, X_{\mathbf{cl}}(s_1; t, x, v), v')}{dv'} \right\} \neq 0. \qquad (35)$$

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Section 4.2 is devoted to the study of the bounce-back reflection. The bounce-back cycles $X_{\mathbf{cl}}(s;t,x,v)$ from a given point (t,x,v) is relatively simple, just going back and forth between two fixed points $x_{\mathbf{b}}(x,v)$ and $x_{\mathbf{b}}(x_{\mathbf{b}}(x,v),-v)$. Now the change of variable (34) and (35) can be established by the study of set $S_x(v)$ in Section 4.2.2.

Section 4.3 is devoted to the study of specular reflection. The specular cycles $X_{\mathbf{cl}}(s;t,x,v)$ reflect repeatedly with the boundary in general, and $\frac{dX_{\mathbf{cl}}(s;s_1,X_{\mathbf{cl}}(s_1;t,x,v),v')}{dv'}$ is very complicated to compute and (35) is extremely difficult to verify, even in a convex domain. This is in part due to the fact that there is no apparent way to analyze $\frac{dX_{\mathbf{cl}}(s;s_1 X_{\mathbf{cl}}(s_1;t,x,v),v')}{dv'}$ inductively with finite bounces. To overcome such a difficulty, in Section 4.3.2, $\det \frac{dv_k}{dv_1}$ can be computed asymptotically in a delicate iterative fashion for special cycles almost tangential to the boundary, which undergo many small bounces near the boundary. It then follows that $\det\{\frac{dX_{\mathbf{cl}}(s;s_1,X_{\mathbf{cl}}(s_1;t,x,v),v')}{dv'}\} \neq 0$ for these special cycles. This crucial observation is then combined with analyticity of ξ and Lemma 8 to conclude that the set of $\det\{\frac{dX_{\mathbf{cl}}(s;X_{\mathbf{cl}}(s_1,x,v),v')}{dv'}\} = 0$ is arbitrarily small (Lemma 34), and the change of variable (34) is almost always valid. The analyticity plays an important role in our proof.

Section 4.4 is devoted to the study of diffuse reflection. The diffuse cycles $X_{\mathbf{cl}}(s;t,x,v)$ contain more and more independent variables and (32) involves their integrations. Similar change of variable (33) is expected with respect to one of such independent variables. However, the main difficulty in this case is the L^∞ control of $G(t)$ which satisfies (29). The most natural L^∞ estimate for $G(t)$ is for the weight $w = \mu^{-\frac{1}{2}}$, in which the diffuse boundary condition takes the form

$$h(t,x,v) = c_\mu \int_{v'\cdot n(x)>0} h(t,x,v')\mu(v')\{v'\cdot n(x)\}dv'$$

with $c_\mu \int_{v'\cdot n(x)>0} \mu(v')\{v'\cdot n(x)\}dv' = 1$. But such a natural (strong) weight $\mu^{-\frac{1}{2}}$ exactly makes the whole linear Boltzmann theory break down (Lemma 7), so we have to use a weaker weight, which is closer to $\mu^{-\frac{1}{2}}$. This new weight will introduce serious new difficulty since no natural maximum principle is available now from the new boundary condition (184). Moreover, for any (t,x,v), since there are always particles moving almost tangential to the boundary in the bounce-back reflection, it is impossible to reach down the initial plane no matter how many cycles the particles take. In other words, there is no explicit expression for $G(t)$ in terms of initial data completely. To establish the L^∞ estimate, we make crucial observation in Lemma 37 that the measure of those particles can not reach initial plane after $k-$bounces is small when k is large. We therefore can obtain an approximate representation formula for $G(t)$ by the initial datum, with only finite number of bounces.

Section 5 is devoted to the proofs of the main nonlinear decay and continuity results of this paper.

Our contribution opens new lines of research about such interesting questions as specular reflections in non-convex domains, decay for the soft potentials,

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higher regularity in a convex domain, characterization of propagation of singularity in a non-convex domain, as well as charged particles interacting with fields. Moreover, our new $L^2 - L^\infty$ theory will shed new lights in other investigations of Boltzmann equation, in which regularity of the solutions is difficult to employ [EGM] [GS].

2 Preliminary

Lemma 5 *Let Ω be strictly convex defined in (4). Define the functional along the trajectories $\frac{dX(s)}{ds} = V(s), \frac{dV(s)}{ds} = 0$ in (6) as:*

$$\alpha(s) \equiv \xi^2(X(s)) + [V(s) \cdot \nabla\xi(X(s))]^2 - 2\{V(s) \cdot \nabla^2\xi(X(s)) \cdot V(s)\}\xi(X(s)). \quad (36)$$

Let $X(s) \in \bar{\Omega}$ for $t_1 \leq s \leq t_2$. Then there exists constant $C_\xi > 0$ such that

$$
\begin{aligned}
e^{C_\xi(|V(t_1)|+1)t_1}\alpha(t_1) &\leq e^{C_\xi(|V(t_1)|+1)t_2}\alpha(t_2); \\
e^{-C_\xi(|V(t_1)|+1)t_1}\alpha(t_1) &\geq e^{-C_\xi(|V(t_1)|+1)t_2}\alpha(t_2).
\end{aligned}
\quad (37)
$$

Proof. Under the convexity assumption (4), we notice that $\alpha(s) \geq 0$ for $t_1 \leq s \leq t_2$. Since $\frac{dV(s)}{ds} \equiv 0$ by (6), we compute the derivative of $\alpha(s)$ in (36) as

$$
\begin{aligned}
\frac{d\alpha(s)}{ds} &= 2\xi(X(s))[\nabla\xi(X(s)) \cdot V(s)] + 2[V(s)\nabla^2\xi(X(s))V(s))][V(s) \cdot \nabla\xi(X(s))] \\
&\quad -2\{V(s) \cdot \nabla^2\xi(X(s)) \cdot V(s)\}[\nabla\xi(X(s)) \cdot V(s)] \\
&\quad -2[V(s) \cdot \nabla^3\xi(X(s))V(s) \cdot V(s)]\xi(X(s)).
\end{aligned}
$$

Note that the second and the third terms on the right hand side cancel exactly. By the convexity (4), there exists $C_\xi > 0$ so that we can further bound the last term by $\alpha(s)$ as:

$$
\begin{aligned}
&|2[V(s) \cdot \nabla^3\xi(X(s))V(s) \cdot V(s)]\xi(X(s))| \\
\leq\ & C_\xi|V(t_1)| \times |\{-2V(s) \cdot \nabla^2\xi(X(s)) \cdot V(s)\}\xi(X(s))| \\
\leq\ & C_\xi|V(t_1)|\alpha(s).
\end{aligned}
$$

We therefore have from the definition (36):

$$
\begin{aligned}
|\frac{d\alpha(s)}{ds}| &\leq \xi^2(X(s)) + [\nabla\xi(X(s) \cdot V(s)]^2 + C_\xi|V(t_1)|\alpha(s) \\
&\leq \{C_\xi|V(t_1)| + 1\}\alpha(s).
\end{aligned}
$$

Our lemma thus follows from the standard Gronwall inequality. ∎

Lemma 6 *Let (t, x, v) be connected with $(t - t_{\mathbf{b}}, x_{\mathbf{b}}, v)$ backward in time through a trajectory of (6).*
(1) The backward exit time $t_{\mathbf{b}}(x, v)$ is lower semicontinuous.

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(2) If

$$v \cdot n(x_{\mathbf{b}}) = v \cdot \frac{\nabla \xi(x_{\mathbf{b}})}{|\nabla \xi(x_{\mathbf{b}})|} < 0, \tag{38}$$

then $(t_{\mathbf{b}}(x, v), x_{\mathbf{b}}(x, v))$ are smooth functions of (x, v) so that

$$\nabla_x t_{\mathbf{b}} = \frac{n(x_{\mathbf{b}})}{v \cdot n(x_{\mathbf{b}})}, \quad \nabla_v t_{\mathbf{b}} = \frac{t_{\mathbf{b}} n(x_{\mathbf{b}})}{v \cdot n(x_{\mathbf{b}})},$$

$$\nabla_x x_{\mathbf{b}} = I + \nabla_x t_{\mathbf{b}} \otimes v, \quad \nabla_v x_{\mathbf{b}} = t_{\mathbf{b}} I + \nabla_v t_{\mathbf{b}} \otimes v. \tag{39}$$

Furthermore, if ξ is real analytic, then $(t_{\mathbf{b}}(x, v), x_{\mathbf{b}}(x, v))$ are also real analytic.

(3) Let $x_i \in \partial\Omega$, for $i = 1, 2$, and let (t_1, x_1, v) and (t_2, x_2, v) be connected with the trajectory $\frac{dX(s)}{ds} = V(s), \frac{dV(s)}{ds} = 0$ which lies inside $\bar{\Omega}$. Then there exists a constant $C_\xi > 0$ such that

$$|t_1 - t_2| \geq \frac{|n(x_1) \cdot v|}{C_\xi |v|^2}. \tag{40}$$

(4) Define the boundary mapping

$$\Phi_{\mathbf{b}} : (t, x, v) \to (t - t_{\mathbf{b}}, x_{\mathbf{b}}(x, v), v) \in \mathbf{R} \times \{\gamma_0 \cup \gamma_-\}. \tag{41}$$

Then $\Phi_{\mathbf{b}}$ and $\Phi_{\mathbf{b}}^{-1}$ maps zero measure sets to zero-measure sets between either $\{t\} \times \Omega \times \mathbf{R}^3$ and $\mathbf{R} \times \{\gamma_0 \cup \gamma_+\}$ or $\mathbf{R} \times \{\gamma_0 \cup \gamma_+\} \to \mathbf{R} \times \{\gamma_0 \cup \gamma_-\}$.

Proof. **(1):** We need to show that the set $\{(x, v) : x \in \Omega$ and $t_{\mathbf{b}}(x, v) > c\}$ is open for all $c \in \mathbf{R}$. Let $t_{\mathbf{b}}(x_0, v_0) > c + \varepsilon$, for some $\varepsilon > 0$ small. From the definition of $t_{\mathbf{b}}(x, v)$ in (7), $x_0 - s v_0 \in \Omega$ for all $0 \leq s \leq c + \varepsilon < t_{\mathbf{b}}(x_0, v_0)$. Since Ω is open, we deduce that for (x, v) close to (x_0, v_0), by continuity, $x - vs \in \Omega$ for all $c \leq s \leq c + \varepsilon$. This implies that $t_{\mathbf{b}}(x, v) > c$. Hence $t_{\mathbf{b}}(x, v)$ is lower semicontinuous.

Proof of (2): Since $x_{\mathbf{b}} \in \partial\Omega$, $\xi(x_{\mathbf{b}}(x, v)) = \xi(x - t_{\mathbf{b}} v) = 0$. But from (38) and the fact $|\nabla\xi(x_{\mathbf{b}})| \neq 0$, we have

$$\partial_{t_{\mathbf{b}}} \xi(x - t_{\mathbf{b}} v)|_{t_{\mathbf{b}}} = -v \cdot \nabla\xi(x_{\mathbf{b}}) > 0.$$

Therefore, by the Implicit Function Theorem, we can solve for smooth $t_{\mathbf{b}}(x, v)$ and deduce (39). Furthermore, when ξ is analytic, so are $t_{\mathbf{b}}$ and $x_{\mathbf{b}}$ by Theorem 15.3 in [D].

Proof of (3): Notice that for $x_1 \in \partial\Omega$,

$$\lim_{\substack{y \to x_1 \\ y \in \partial\Omega}} \frac{|\{x_1 - y\} \cdot n(x_1)|}{|x_1 - y|} = 0.$$

Hence, we have $|\{x_1 - y\} \cdot n(x_1)| \leq C_\xi |x_1 - y|^2$ for all $y \in \partial\Omega$. Taking inner product of $x_1 - x_2 = v(t_1 - t_2)$ with $n(x_1)$, we get

$$C_\xi |v|^2 |t_1 - t_2|^2 \geq C_\xi |x_1 - x_2|^2 \geq |\{x_1 - x_2\} \cdot n(x_1)| = |v(t_1 - t_2) \cdot n(x_1)|.$$

We thus deduce (40) by dividing $|t_1 - t_2|$.

Proof of (4): We define a map from $\mathbf{R} \times \partial\Omega \times \mathbf{R}^3$ to $\{t\} \times \mathbf{R}^3 \times \mathbf{R}^3$ as

$$\Psi_t : (t', x', v') = (t, x' + v'(t - t'), \ v'). \tag{42}$$

Recall the boundary map $\Phi_{\mathbf{b}}(t, x, v)$ in (41). From the definition of $t_{\mathbf{b}}$ in (7), $\Phi_{\mathbf{b}}$ is one to one from either from $\{t\} \times \Omega \times \mathbf{R}^3$ or from $\mathbf{R} \times \{\gamma_0 \cup \gamma_+\}$ to $\mathbf{R} \times \{\gamma_0 \cup \gamma_-\}$. Denote its inverse by $\Phi_{\mathbf{b}}^{-1}$.

In the case that $v \cdot n(x_{\mathbf{b}}) = 0$, i.e., $\Phi_{\mathbf{b}}(t, x, v) \in \gamma_0$, the grazing set, then $(t, x, v) \in \Psi_t(\gamma_0)$. In (42), γ_0 is characterized by the five-dimensional space: $x' \in \partial\Omega$, $v' \cdot n(x') = 0$, $t' \in \mathbf{R}$. Since Ψ_t is a smooth map, $\Psi_t(\gamma_0)$ is also a five-dimensional space locally at $(x', v', t') \in \gamma_0$. This implies that $\Phi_{\mathbf{b}}^{-1}(\gamma^0) \subset \Psi_t(\gamma_0)$ is a zero-measure set in $\{t\} \times \Omega \times \mathbf{R}^3$.

In the case that $v \cdot n(x_{\mathbf{b}}) \neq 0$, we consider the map Ψ_t where $\xi(x') = 0$. We may assume that $\partial_{x_1}\xi(x') \neq 0$, and $x_1' = \eta(x_2', x_3')$ with some smooth function η. Now $\Psi_t = (t, \eta(x_2', x_3') + v_1'(t - t'), x_2' + v_2'(t - t'), x_3' + v_3'(t - t'), v')$ and we compute the Jacobian matrix of Ψ_t for (t', x_2', x_3', v') to get

$$\begin{pmatrix} -v_1' & \partial_{x_2'}\eta & \partial_{x_3'}\eta & t - t' & 0 & 0 \\ -v_2' & 1 & 0 & 0 & t - t' & 0 \\ -v_3' & 0 & 1 & 0 & 0 & t - t' \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and its determinant is exactly $\pm \{v' \cdot n(x')\} \sqrt{1 + |\nabla\eta|^2} \neq 0$ if $(x', v') \notin \gamma_0$. Hence, locally, Ψ_t is a smooth homeomorphism preserving zero-measure sets away from γ_0. Notice that from the uniqueness in part (2) of Lemma 6, $\Phi_{\mathbf{b}}^{-1} = \Psi_t$ and $\Phi_{\mathbf{b}} = \Psi_t^{-1}$ locally if $v \cdot n(x_{\mathbf{b}}) \neq 0$. Hence, for any $k \geq 1$, by a finite covering for a compact set, $\Phi_{\mathbf{b}}^{-1}$ preserves the zero-measure sets on

$$A_k \equiv \{(t', x', v') | \ x' \in \partial\Omega, |v' \cdot n(x')| \geq \frac{1}{k}, |v'| \leq k, |t'| \leq k\} \cap \Phi_{\mathbf{b}}(\{t\} \times \Omega \times \mathbf{R}^3).$$

To prove $\Phi_{\mathbf{b}}$ preserves the zero-measure sets between $\{t\} \times \Omega \times \mathbf{R}^3$ and $\mathbf{R} \times \{\gamma_0 \cup \gamma_+\}$, we take any set $S \in \mathbf{R} \times \{\gamma_0 \cup \gamma_+\}$ with $|S| = 0$. Clearly, $S = \{\mathbf{R} \times \gamma_0 \cap S\} \cup_{k=1}^\infty \{A_k \cap S\}$. Therefore, $\Phi_{\mathbf{b}}^{-1}(S) \subset \Psi_t(\gamma_0) \cup_{k=1}^\infty \Phi_{\mathbf{b}}^{-1}(A_k \cap S)$, and $|\Phi_{\mathbf{b}}^{-1}(A_k \cap S)| = 0$ and $|\Psi_t(\gamma_0)| = 0$. On the other hand, if $S \in \{t\} \times \Omega \times \mathbf{R}^3$, we have $\Phi_{\mathbf{b}}(S) = \{\mathbf{R} \times \gamma_0 \cap \Phi_{\mathbf{b}}(S)\} \cup_{k=1}^\infty \{A_k \cap \Phi_{\mathbf{b}}(S)\}$. If $|S| = 0$, then $|A_k \cap \Phi_{\mathbf{b}}(S)| = 0$, because $\Phi_{\mathbf{b}}^{-1}\{A_k \cap \Phi_{\mathbf{b}}(S)\} \subset S$ has measure zero and $\Phi_{\mathbf{b}} = \Psi_t^{-1}$ maps zero-measure sets to zero-measure sets.

To prove $\Phi_{\mathbf{b}}$ preserves the zero-measure sets from $\mathbf{R} \times \{\gamma_0 \cup \gamma_+\} \to \mathbf{R} \times \{\gamma_0 \cup \gamma_-\}$, we take $S \in \mathbf{R} \times \{\gamma_0 \cup \gamma_+\}$ with $|S| = 0$. Consider the set $\Phi_{\mathbf{b}}^{-1}\{\Phi_{\mathbf{b}}(S) \backslash \gamma_0\}$. For any point $(t, x, v) \in \Phi_{\mathbf{b}}^{-1}\{\Phi_{\mathbf{b}}(S) \setminus \gamma_0\}$, we know that $v \cdot n(x_{\mathbf{b}}) \neq 0$. This implies from (40) that $t_{\mathbf{b}} > 0$ and $t - t_{\mathbf{b}} < t$. We can choose a fixed s between t and $t - t_{\mathbf{b}}$. Locally around (t, x, v),

$$\Phi_{\mathbf{b}} = \Phi_{\mathbf{b}}(s) \circ \Psi_s. \tag{43}$$

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where Ψ_s defined in (42) which maps $\mathbf{R} \times \{\gamma_0 \cup \gamma_+\}$ to the plane $\{s\} \times \Omega \times \mathbf{R}^3$, and $\Phi_{\mathbf{b}}(s)$ maps $\{s\} \times \Omega \times \mathbf{R}^3$ to $\mathbf{R} \times \{\gamma_0 \cup \gamma_-\}$. Since $\Phi_{\mathbf{b}}$ is one to one, we have $\Phi_{\mathbf{b}}^{-1}\{\Phi_{\mathbf{b}}(S) \setminus \gamma_0\} \subset S$, so that $|\Phi_{\mathbf{b}}^{-1}\{\Phi_{\mathbf{b}}(S) \setminus \gamma_0\}| = 0$. Therefore, from previous arguments, Ψ_s preserves zero-measure sets from $\mathbf{R} \times \{\gamma_0 \cup \gamma_+\}$ to $\{s\} \times \Omega \times \mathbf{R}^3$, while $\Phi_{\mathbf{b}}(s)$ preserves zero-measure sets from $\{s\} \times \Omega \times \mathbf{R}^3$ set to $\mathbf{R} \times \{\gamma_0 \cup \gamma_-\}$. Hence, by (43), $|\Phi_{\mathbf{b}}(S) \setminus \gamma_0| = |\Phi_{\mathbf{b}}(s) \circ \Psi_s[\Phi_{\mathbf{b}}^{-1}\{\Phi_{\mathbf{b}}(S) \setminus \gamma_0\}]| = 0$. We therefore deduce $|\Phi_{\mathbf{b}}(S)| = 0$, for S inside a neighborhood of (t, x, v).

On the other hand, we take $S \in \mathbf{R} \times \{\gamma_0 \cup \gamma_-\}$ with $|S| = 0$ and we need to show $|\Phi_{\mathbf{b}}^{-1}(S)| = 0$. For any point $(t, x, v) \in \Phi_{\mathbf{b}}^{-1}(S) \setminus \gamma_0$, $v \cdot n(x) \neq 0$, so that if $(t', x', v) = \Phi_{\mathbf{b}}(t, x, v)$, then $t' < t$. Hence, we again can find $t' < s < t$ such that, near (t, x, v), $\Phi_{\mathbf{b}}^{-1} = \Psi_s^{-1} \circ \Phi_{\mathbf{b}}^{-1}(s)$ from (43). Since $|\Phi_{\mathbf{b}}\{\Phi_{\mathbf{b}}^{-1}(S) \setminus \gamma_0\}| \leq |S| = 0$, for $\Phi_{\mathbf{b}}^{-1}(S)$ near (t, x, v), $\{\Phi_{\mathbf{b}}^{-1}(S) \setminus \gamma_0\} = \Psi_s^{-1} \circ \Phi_{\mathbf{b}}^{-1}(s)[\Phi_{\mathbf{b}}\{\Phi_{\mathbf{b}}^{-1}(S) \setminus \gamma_0\}]$. It follows (43) again that $|\Phi_{\mathbf{b}}^{-1}(S) \setminus \gamma_0| = 0$, $|\Phi_{\mathbf{b}}^{-1}(S)| = 0$ for S inside a neighborhood of $(t - t_{\mathbf{b}}, x_{\mathbf{b}}, v)$. The general case follows from a countable covering for $\mathbf{R} \times \{\gamma_0 \cup \gamma_+\}$ and $\{s\} \times \Omega \times \mathbf{R}^3$. ■

Lemma 7 *Recall (9) and the Grad estimate [Gr2] for hard potentials:*

$$|K(v, v')| \leq C\{|v - v'| + |v - v'|^{-1}\} e^{-\frac{1}{8}|v - v'|^2 - \frac{1}{8}\frac{||v|^2 - |v'|^2|^2}{|v - v'|^2}}. \qquad (44)$$

Recall w in (20). Then there exists $0 \leq \varepsilon(\theta) < 1$ and $C_\theta > 0$ such that for $0 \leq \varepsilon < \varepsilon(\theta)$,

$$\int \{|v - v'| + |v - v'|^{-1}\} e^{-\frac{1-\varepsilon}{8}|v - v'|^2 - \frac{1-\varepsilon}{8}\frac{||v|^2 - |v'|^2|^2}{|v - v'|^2}} \frac{w(v)}{w(v')} dv' \leq \frac{C}{1 + |v|}. \qquad (45)$$

Proof. By (20), we first notice that for some $C_{\rho, \beta} > 0$,

$$\left| \frac{w(v)}{w(v')} \right| \leq C[1 + |v - v'|^2]^{|\beta|} e^{-\theta\{|v'|^2 - |v|^2\}}.$$

Let $v - v' = \eta$ and $v' = v - \eta$ in the integral of (45). By (44), we now compute the total exponent in $K(v, v')\frac{w(v)}{w(v')}$ as:

$$-\frac{1}{8}|\eta|^2 - \frac{1}{8}\frac{||\eta|^2 - 2v \cdot \eta|^2}{|\eta|^2} - \theta\{|v - \eta|^2 - |v|^2\}$$

$$= -\frac{1}{4}|\eta|^2 + \frac{1}{2}v \cdot \eta - \frac{1}{2}\frac{|v \cdot \eta|^2}{|\eta|^2} - \theta\{|\eta|^2 - 2v \cdot \eta\}$$

$$= (-\theta - \frac{1}{4})|\eta|^2 + (\frac{1}{2} + 2\theta)v \cdot \eta - \frac{1}{2}\frac{\{v \cdot \eta\}^2}{|\eta|^2}.$$

Since $\theta < \frac{1}{4}$, the discriminant of the above quadratic form of $|\eta|$ and $\frac{v \cdot \eta}{|\eta|}$ is

$$\Delta = (\frac{1}{2} + 2\theta)^2 + 2(-\theta - \frac{1}{4}) = 4\theta^2 - \frac{1}{4} < 0.$$

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Hence, the quadratic form is negative definite. We thus have, for $\varepsilon > 0$ sufficiently small, there is $C_\theta > 0$ such that the following perturbed quadratic form is still negative definite

$$-\frac{1-\varepsilon}{8}|\eta|^2 - \frac{1-\varepsilon}{8}\frac{||\eta|^2 - 2v\cdot\eta|^2}{|\eta|^2} - \theta\{|\eta|^2 - 2v\cdot\eta\}$$

$$\leq -C_\theta\{|\eta|^2 + \frac{|v\cdot\eta|^2}{|\eta|^2}\} = -C_\theta\{\frac{|\eta|^2}{2} + (\frac{|\eta|^2}{2} + \frac{|v\cdot\eta|^2}{|\eta|^2})\}$$

$$\leq -C_\theta\{\frac{|\eta|^2}{2} + |v\cdot\eta|\}. \tag{46}$$

Therefore, for given $|v| \geq 1$, we make another change of variable $\eta_{||} = \{\eta\cdot\frac{v}{|v|}\}\frac{v}{|v|}$, and $\eta_\perp = \eta - \eta_{||}$ so that $|v\cdot\eta| = |v||\eta_{||}|$ and $|v - v'| \geq |\eta_\perp|$. We can absorb $\{1 + |\eta|^2\}^{|\beta|}$, $|\eta|\{1 + |\eta|^2\}^{|\beta|}$ by $e^{\frac{C_\theta}{4}|\eta|^2}$, and bound the integral in (45) by (46):

$$C_\beta \int_{\mathbf{R}^2} (\frac{1}{|\eta_\perp|} + 1)e^{-\frac{C_\theta}{4}|\eta|^2}\left\{\int_{-\infty}^{\infty} e^{-C_\theta|v|\times|\eta_{||}|}d|\eta_{||}|\right\} d\eta_\perp$$

$$\leq \frac{C_\beta}{|v|}\int_{\mathbf{R}^2}(\frac{1}{|\eta_\perp|} + 1)e^{-\frac{C_\theta}{4}|\eta_\perp|^2}\left\{\int_{-\infty}^{\infty}e^{-C_\theta|y|}dy\right\} d\eta_\perp \quad (y = |v|\times|\eta_{||}|).$$

We thus deduce our lemma since both integrals are finite. ∎

Lemma 8 *Let $\kappa(y)$ be a real analytic function of $y \in \mathbf{R}^n$ in a region such that $\kappa(y)$ is not identically zero. Then the set $\{y : \kappa(y) = 0\}$ has zero $n-$dimensional Lebesque measure.*

Proof. We use an induction on the dimension n. When $n = 1$, we assume $\kappa(y^0) = 0$. Since κ is analytic, for y near y^0, we have

$$\kappa(y) = \kappa(y^0) + \sum_{k=1}^{\infty}\frac{\kappa^{(k)}(y^0)}{k!}(y - y^0)^k.$$

Since $\kappa(y)$ is not identically zero, we can always assume a smallest k_1 such that $\frac{\kappa^{(k_1)}(y^0)}{k_1!} \neq 0$. We therefore can rewrite

$$\kappa(y) = (y - y^0)^{k_1} \times \left\{\sum_{k\geq k_1}^{\infty}\frac{\kappa^{(k)}(y^0)}{k!}(y - y^0)^{k-k_1}\right\}.$$

Hence $\kappa(y) = 0$ for $y - y^0$ sufficiently small implies $y = y^0$ (an isolated point), which has zero one dimensional measure.

Assume that the lemma is valid for m. For $m+1$ dimensional case, we assume $\kappa(y^0) = 0$. We first notice that by finite open coverings for any compact subset, it suffices to show that for any y^0 such that $\kappa(y^0) = 0$, then there is a ball $\{y : |y - y^0| < \delta\}$ such that $|\{y : |y - y^0| < \delta, \kappa(y) = 0\}| = 0$.

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Now for any $y \neq y^0$ and $|y - y^0| < \delta$, since $\kappa(y)$ is real analytic, we have

$$\kappa(y) = \kappa(y_0) + \sum_{|k|=1}^{\infty} \frac{\kappa^{(k)}(y^0)}{k!}(y - y^0)^k$$

where the multi-index $k = [k_1, k_2, ..., k_m]$, $k! = k_1! k_2! ... k_m!$, while $(y - y^0)^k = \Pi_{l=1}^{m}(y_l - y_l^0)^{k_l}$. Since $\kappa(y)$ is not identically zero, there exists $\bar{k}$ such that $\kappa^{(\bar{k})}(y^0) \neq 0$. Without loss of generality, we can further assume that $\bar{k}_1 \neq 0$, so that $(y - y^0)^{k_1}$ contains the factor $(y_1 - y_1^0)^{\bar{k}_1}$. Furthermore, we can assume $\bar{k}_1 \geq 0$ is the smallest among those non-zero terms, so that every term $(y - y^0)^k$ contains the common factor $(y_1 - y_1^0)^{\bar{k}_1}$. We therefore can rewrite:

$$\kappa(y) = (y_1 - y_1^0)^{\bar{k}_1} \left\{ \sum \frac{\kappa^{(k)}(y^0)}{k!}(y - y^0)^{k - \bar{k}_1} \right\}.$$

For $y_1 \neq y_1^0$, $\kappa(y) = 0$ implies

$$\kappa_1(y) \equiv \sum \frac{\kappa^{(k)}(y^0)}{k!}(y - y^0)^{k - \bar{k}_1} = 0. \tag{47}$$

Clearly, for any given y_1, $\kappa_1(y_1, y_2, ..., y_{m+1})$ is an analytical function for m variables $\tilde{y} = [y_2, ..., y_{m+1}]$. Therefore, for fixed y_1, we can expand $\kappa_1(y_1, \tilde{y})$ around $[y_2^0, ..., y_{m+1}^0]$ to get

$$\kappa_1(y_1, \tilde{y}) \equiv \sum_{k=[0, k_2, k_3, ..., k_{m+1}]} \frac{\kappa_1^{(k)}(y_1, y_2^0, ..., y_{m+1}^0)}{k!}(\tilde{y} - \tilde{y}^0)^k.$$

Since by our choices of $\bar{k}$ and $\bar{k}_1$, the multi-index $\bar{k} - \bar{k}_1 = [0, \bar{k}_2, ..., \bar{k}_{m+1}]$, and we can consider the term

$$\frac{\kappa_1^{(\bar{k} - \bar{k}_1)}(y_1, y_2^0, ..., y_{m+1}^0)}{(\bar{k} - \bar{k}_1)!}(\tilde{y} - \tilde{y}^0)^{\bar{k} - \bar{k}_1}.$$

We compute $\kappa_1^{(\bar{k} - \bar{k}_1)}$ from (47) as

$$\kappa_1^{(\bar{k} - \bar{k}_1)}(y_1, y_2^0, ..., y_{m+1}^0)|_{y_1 = y_1^0} = \sum \frac{\kappa^{(k)}(y^0)}{k!} \partial^{\bar{k} - \bar{k}_1} \{(y - y^0)^{k - \bar{k}_1}\} = \frac{(\bar{k} - \bar{k}_1)! \kappa^{(\bar{k})}(y^0)}{\bar{k}!} \neq 0,$$

by our choices of $\bar{k}$ and $\bar{k}_1$. From the continuity of $\kappa_1^{(\bar{k} - \bar{k}_1)}(y_1, y_2^0, ..., y_{m+1}^0)$ with respect to y_1, for $|y_1 - y_1^0| < \delta$ small, we deduce that

$$\kappa_1^{(\bar{k} - \bar{k}_1)}(y_1, y_2^0, ..., y_{m+1}^0) \neq 0.$$

Therefore, $\kappa_1(y_1, y_2, ..., y_{m+1})$ is an analytical function which is not identically zero for all $|y - y^0| < \delta$ sufficiently small. By the induction hypothesis, for

any fixed y_1, $\kappa_1(y_1, y_2, ..., y_{m+1}) = 0$ is a set of $m-$dimensional Lebesque zero measure set for $y_2, y_3, ... y_{m+1}$. We now apply the Fubini theorem to compute

$$
\begin{aligned}
|\{\kappa_1(y) &= 0, |y - y^0| < \delta\}| = \int_{\mathbf{R}^{m+1}} \mathbf{1}_{\{\kappa_1(y)=0\}}(y_1, y_2, ..., y_{m+1}) \mathbf{1}_{|y-y^0|<\delta} \\
&= \int_{\mathbf{R}} \left(\int_{\mathbf{R}^m} \mathbf{1}_{\{\kappa_1(y)=0\}}(y_1, y_2, ..., y_{m+1}) \mathbf{1}_{|y-y^0|<\delta} dy_2 dy_3 ... dy_{m+1} \right) dy_1 \\
&\leq \int_{\mathbf{R}} |\{(y_2, y_3, ..., y_{m+1}) : \kappa_1(y_1, y_2, ..., y_{m+1}) = 0, \; |y - y^0| < \delta\}| dy_1 \\
&= 0.
\end{aligned}
$$

Therefore, inside $|y - y^0| < \delta$ for δ sufficiently small, we have

$$
\{y : \kappa(y) = 0\} \subset \{y_1 = y_1^0\} \cup \{\kappa_1(y) = 0\},
$$

and both sets have zero $(m+1)-$ dimensional Lebesque measure. ∎

Lemma 9 *Recall (20) and (10). We have*

$$
|w\Gamma[g_1, g_2](v)| \leq C\{w(v)(|v| + 1)^\gamma |g_1(v)| + ||wg_1||_\infty\} ||wg_2||_\infty.
$$

Proof. First consider the second term Γ_{loss} in (10). We have

$$
\int_{\mathbf{R}^3} |u - v|^\gamma |\mu^{1/2}(u) g_2(x, u)| du \leq C\{|v| + 1\}^\gamma ||wg_2||_\infty,
$$

Hence $w\Gamma_{\text{loss}}[g_1, g_2]$ is bounded by

$$
w|g_1| \int |u - v|^\gamma |\mu^{1/2}(u) g_2(x, u)| du \leq Cw(v)\{|v| + 1\}^\gamma |g_1(v)| \times ||wg_2||_\infty.
$$

For Γ_{gain} in (10), by $|u'|^2 + |v'|^2 = |u|^2 + |v|^2$, $w(v) \leq Cw(v')w(u')$, and

$$
\int q_0(\theta) |u - v|^\gamma e^{-|u-v|^2/4} w(v) |g_1(u') g_2(v')| d\omega du
$$

$$
\leq \int q_0(\theta) |u - v|^\gamma e^{-|u-v|^2/4} w(u') w(v') |g_1(u') g_2(v')| d\omega du
$$

$$
\leq ||wg_1||_\infty \times ||w_2 g_2||_\infty \int |u - v|^\gamma e^{-|u-v|^2/4} du.
$$

Since $0 \leq \gamma \leq 1$, this completes the proof. ∎

3 L^2 Decay Theory

We define the boundary integration for $g(x, v)$, $x \in \partial\Omega$,

$$
\int_{\gamma_\pm} g d\gamma = \int_{\gamma_\pm} g(x, v) |n_x \cdot v| dS_x dv \tag{48}
$$

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where dS_x is the standard surface measure on $\partial\Omega$. We also define $||h||_\gamma = ||h||_{\gamma_+} + ||h||_{\gamma_-}$ to be the $L^2(\gamma)$ with respect to the measure $|n_x \cdot v|dS_x dv$. For fixed $x \in \partial\Omega$, denote the boundary inner product over $\gamma_\pm$ in v as

$$\langle g_1, g_2 \rangle_{\gamma_\pm}(t, x) = \int_{\pm v \cdot n(x) > 0} g_1(t, x, v)g_2(t, x, v)|n(x) \cdot v|dv.$$

By (15), we also define a different L_v^2-projection for any boundary function $g(x, v)$ toward the unit vector $\sqrt{c_\mu \mu(v)}$ with respect to $\langle \cdot, \cdot \rangle$ as:

$$P_\gamma g = \{\int_{n(x) \cdot v > 0} g(t, x, v)\sqrt{\mu(v)}n(x) \cdot v dv\}c_\mu \sqrt{\mu(v)}. \qquad (49)$$

Our main theorem of this section is

Theorem 10 *Let $f(t, x, v) \in L^2$ be the (unique) solution to the linear Boltzmann equation (23) with trace $f_\gamma \in L_{loc}^2 (\mathbf{R}_+; L^2(\gamma))$.*

(1) If f satisfies the in-flow boundary condition (11), then there exists $\lambda > 0$ and $C > 0$ such that for all $0 \le t \le \infty$,

$$e^{2\lambda t}||f(t)||^2 \le 2\{||f(0)||^2 + \int_0^t e^{2\lambda s}||g(s)||_{\gamma_-}^2 ds\}.$$

(2) Let f satisfy the bounce-back boundary condition (12), then there exists $\lambda > 0$ and $C > 0$ such that $\sup_{0 \le t \le \infty}\{e^{2\lambda t}||f(t)||^2\} \le 2||f(0)||^2$.

(3) Let f satisfy the specular reflection condition (14), and the mass and energy conservation laws (17) and (18). In the case Ω has any axis of rotational symmetry (5), we further require that the corresponding conservation of the angular momentum (19). Then there exists $\lambda > 0$ and $C > 0$ such that $\sup_{0 \le t \le \infty}\{e^{2\lambda t}||f(t)||^2\} \le 2||f(0)||^2$.

(4) If f satisfies the diffusive boundary condition (16) and the mass conservation (17), then there exists $\lambda > 0$ and $C > 0$ such that $\sup_{0 \le t \le \infty}\{e^{2\lambda t}||f(t)||^2\} \le 2||f(0)||^2$.

We remark that the existence of such a L^2 solution f with its trace $f_\gamma \in L_{\text{loc}}^2 (\mathbf{R}_+; L^2(\gamma))$ (which guarantees the uniqueness) is in general not known for the bounce-back and specular boundary conditions within L^2 framework. This is due to the possible blow-up of $L_{\text{loc}}^2 (\mathbf{R}_+; L^2(\gamma))$ at the grazing set γ_0. See [BP], [CIP] and [U1] for more details. On the other hand, $\int_0^t ||f(s)||_\gamma^2 ds < \infty$ will be established by the study of (27) in Theorems 28, 35 and 41 with property $h = wf \in L^\infty$ and $h = wf \in L^\infty(\gamma)$. We mainly will establish the following:

Proposition 11 *(1) There exists $M > 0$ such that for any solution $f(t, x, v)$ to the linearized Boltzmann equation (23),*

$$\int_0^1 ||\mathbf{P}f(s)||_\nu^2 ds \le M\{\int_0^1 ||(\mathbf{I} - \mathbf{P})f(s)||_\nu^2 ds + \int_0^1 ||f(s)||_\gamma^2 ds\}. \qquad (50)$$

(2) There exists $M > 0$ such that for any solution $f(t, x, v)$ to the linearized Boltzmann equation (23) satisfying the bounce-back boundary condition (12) and the mass-energy conservation laws (17) and (18), we have

$$\int_0^1 ||\mathbf{P}f(s)||_\nu^2 ds \le M \int_0^1 ||(\mathbf{I} - \mathbf{P})f(s)||_\nu^2 ds. \tag{51}$$

(3) There exists $M > 0$ such that for any solution $f(t, x, v)$ to the linearized Boltzmann equation (23) satisfying the specular reflection condition (14) and the mass-energy conservation laws (17) and (18), (in the case Ω has any axis of rotational symmetry (5), we further assume conservation of the angular momentum (19)), estimate (51) is valid.

(4) There exists $M > 0$ such that for any solution $f(t, x, v)$ solution to the linearized Boltzmann equation (23) satisfying the diffusive boundary condition (16) and the mass conservation (17),

$$\int_0^1 ||\mathbf{P}f(s)||_\nu^2 ds \le M\{\int_0^1 ||(\mathbf{I} - \mathbf{P})f(s)||_\nu^2 ds + \int_0^1 ||\{I - P_\gamma\}f(s)||_{\gamma_+}^2 ds\}. \tag{52}$$

We first show that Proposition 11 implies Theorem 10.

Proof. of Theorem 10: For any solution f to the linear Boltzmann equation (23), $e^{\lambda t}f(t)$ satisfies

$$\{\partial_t + v \cdot \nabla_x + L\}\{e^{\lambda t}f\} - \lambda e^{\lambda t}f = 0. \tag{53}$$

Let $0 \le N \le t \le N + 1$, N being an integer. We split $[0, t] = [0, N] \cup [N, t]$.

For the time interval $[N, t]$, since $f_\gamma \in L_{\text{loc}}^2 (\mathbf{R}_+; L^2(\gamma))$, $\int_N^t ||f(s)||_\gamma^2 ds < \infty$. We then establish the L^2 energy estimate for $[N, t]$ as

$$||f(t)||^2 + \int_N^t (Lf, f)ds + \int_N^t \int_{\gamma_+} f^2(s)d\gamma ds = ||f(N)||^2 + \int_N^t \int_{\gamma_-} f^2(s)d\gamma ds. \tag{54}$$

For the time interval $[0, N]$, (we may assume $N \ge 1$), since $f_\gamma \in L_{\text{loc}}^2 (\mathbf{R}_+; L^2(\gamma))$, $\int_0^N ||f(s)||_\gamma^2 ds < \infty$. We multiply $e^{\lambda t}f$ with (53) and take L^2 energy estimate over $0 \le s \le N$:

$$e^{2\lambda N}||f(N)||^2 + \int_0^N e^{2\lambda s}(Lf, f)ds - \lambda \int_0^N e^{2\lambda s}||f(s)||^2 ds$$

$$= ||f(0)||^2 + \int_0^N \int_{\gamma_-} e^{2\lambda s}f^2(s)d\gamma ds - \int_0^N \int_{\gamma_+} e^{2\lambda s}f^2(s)d\gamma ds.$$

Dividing the time interval into $\cup_{k=0}^{N-1}[k, k + 1)$ and letting $f_k(s, x, v) \equiv f(k + s, x, v)$ for $k = 0, 1, 2...N - 1$, we deduce

$$e^{2\lambda N}||f(N)||^2 + \sum_{k=0}^{N-1} \int_0^1 \left\{ e^{2\lambda\{k+s\}}(Lf_k, f_k) - \lambda e^{2\lambda\{k+s\}}||f_k(s)||^2 \right\} ds \tag{55}$$

$$= ||f(0)||^2 + \sum_{k=0}^{N-1} \left\{ \int_0^1 \int_{\gamma_-} e^{2\lambda\{k+s\}} f_k^2(s)d\gamma ds - \int_0^1 \int_{\gamma_+} e^{2\lambda\{k+s\}} f_k^2(s)d\gamma ds \right\}.$$

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Notice that $f_k(k+s, x, v)$ satisfies the same linearized Boltzmann equation (23) for $0 \leq s \leq 1$.

In-flow boundary condition (11): Multiplying $\delta_0 e^{2\lambda k}$ with (50) to each $f_k(s, x, v)$ and then summing up over k yields

$$\frac{\delta_0}{2} \sum_{k=0}^{N-1} \{e^{2\lambda k} \int_0^1 ||\{I-P\}f_k||_\nu^2 ds + e^{2\lambda k} \int_0^1 \int_\gamma f_k^2(s) d\gamma ds\} \geq \frac{\delta_0}{2M} \sum_{k=0}^{N-1} e^{2\lambda k} \int_0^1 ||Pf_k||_\nu^2 ds.$$
(56)

Note that $\int_\gamma = \int_{\gamma_+} + \int_{\gamma_-}$, is the total boundary integration. Since

$$(Lf_k, f_k) \geq \delta_0 ||\{I - P\}f_k||_\nu^2 = \frac{\delta_0}{2}||\{I - P\}f_k||_\nu^2 + \frac{\delta_0}{2}||\{I - P\}f_k||_\nu^2$$

and $e^{2\lambda ks} \geq 1$, we apply (56) to the first copy of $\frac{\delta_0}{2}||\{I - P\}f_k||_\nu^2$ in (55), and move the boundary integrals in (56) to the right hand side of (55). Hence,

$$e^{2\lambda N}||f(N)||^2 + \frac{\delta_0}{2M} \sum_{k=0}^{N-1} \int_0^1 e^{2\lambda k}||Pf_k||_\nu^2 ds + \frac{\delta_0}{2} \sum_{k=0}^{N-1} \int_0^1 e^{2\lambda k}||\{I - P\}f_k||_\nu^2 ds$$

$$-C_\nu \lambda \sum_{k=0}^{N-1} \int_0^1 e^{2\lambda\{k+s\}}||f_k(s)||_\nu^2 ds$$

$$\leq ||f(0)||^2 + (1 + \frac{\delta_0}{2}) \sum_{k=0}^{N-1} \int_0^1 \int_{\gamma_-} e^{2\lambda\{k+s\}} g_k^2(s) d\gamma ds - (1 - \frac{\delta_0}{2}) \sum_{k=0}^{N-1} \int_0^1 \int_{\gamma_+} e^{2\lambda\{k+s\}} f_k^2(s) d\gamma ds.$$

Here we have used the fact $|| \cdot || \leq C_\nu || \cdot ||_\nu$ for hard potentials, and the in-flow boundary condition $f_k = g_k$ on γ_-. Combining Pf_k with $\{I - P\}f_k$, and note $e^{2\lambda k} = e^{2\lambda\{k+s\}}e^{-2\lambda s} \geq e^{2\lambda\{k+s\}}e^{-2\lambda}$, we obtain a positive lower bound in the left hand side:

$$\left(\min\{\frac{\delta_0}{2}, \frac{\delta_0}{2M}\}e^{-2\lambda} - C_\nu\lambda\right) \sum_{k=0}^{N-1} \int_0^1 e^{2\lambda\{k+s\}}||f_k(s)||_\nu^2 ds > 0$$

for $C_\nu\lambda < \min\{\frac{\delta_0}{4}, \frac{\delta_0}{4M}\}e^{-2\lambda}$. Changing back to $f_k(t) = f(t + k)$ and letting $1 - \frac{\delta_0}{2} > 0$, we deduce

$$e^{2\lambda N}||f(N)||^2 \leq ||f(0)||^2 + (1 + \frac{\delta_0}{2}) \int_0^N \int_{\gamma_-} e^{2\lambda s} g^2(s) d\gamma ds. \qquad (57)$$

Notice that $e^{2\lambda t} \leq e^{2\lambda\{t-N\}}e^{2\lambda s}$ for $s \geq N$, and since $t \leq N + 1$, we can choose for δ_0 and λ small such that $e^{2\lambda(t-N)}(1 + \frac{\delta_0}{2}) \leq 2$. Hence, multiplying

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$e^{2\lambda t}$ with (54) and combining with (57) yields

$$e^{2\lambda t}||f(t)||^2 + e^{2\lambda t}\int_N^t\int_{\gamma_+} f^2(s)d\gamma ds \le e^{2\lambda t}||f(N)||^2 + e^{2\lambda t}\int_N^t\int_{\gamma_-} g^2(s)d\gamma ds$$

$$\le\ e^{2\lambda\{t-N\}}\{||f(0)||^2 + (1+\frac{\delta_0}{2})\int_0^N\int_{\gamma_-} e^{2\lambda s}g^2(s)d\gamma ds\}$$

$$+e^{2\lambda\{t-N\}}\int_N^t\int_{\gamma_-} e^{2\lambda s}g^2(s)d\gamma ds$$

$$\le\ 2\{||f(0)||^2 + \int_0^t\int_{\gamma_-} e^{2\lambda s}g^2(s)d\gamma ds\}.$$

Bounce-back and specular reflections (12) and (14). In both cases, the total boundary contribution in (55) vanishes:

$$\sum_{k=0}^{N-1}\int_0^1\int_{\gamma_-} e^{2\lambda\{k+s\}} f_k^2(s)d\gamma ds - \sum_{k=0}^{N-1}\int_0^1\int_{\gamma_+} e^{2\lambda\{k+s\}} f_k^2(s)d\gamma ds = 0. \quad (58)$$

For $N \le t < N+1$, we use the same procedure as in the in-flow case (55) and the positivity (51) to get

$$e^{2\lambda N}||f(N)||^2 + \left(\min\{\frac{\delta_0}{2},\frac{\delta_0}{2M}\}e^{-2\lambda} - C_\nu\lambda\right)\sum_{k=0}^{N-1}\int_0^1 e^{2\lambda\{k+s\}}||f_k(s)||_\nu^2 ds \le ||f(0)||^2.$$

For $C_\nu\lambda = \min\{\frac{\delta_0}{4},\frac{\delta_0}{4M}\}e^{-2\lambda} > 0$, changing back to the original $f(k+s)$ leads to

$$e^{2\lambda N}||f(N)||^2 \le ||f(0)||^2. \quad (59)$$

By (54) and (58), $||f(t)|| \le ||f(N)||$. We deduce if $e^{2\lambda(t-N)} \le 2$

$$e^{2\lambda t}||f(t)||^2 \le e^{2\lambda t}||f(N)||^2 \le e^{2\lambda(t-N)}||f(0)||^2 \le 2||f(0)||^2.$$

Diffuse boundary reflection (16). We note from (16) and (49) that $\int_{\gamma_-} f^2(s)d\gamma = \int_{\gamma_+}[P_\gamma f(s)]^2 d\gamma$, so that the boundary contribution in (55) is

$$\int_0^1\int_{\gamma_-} e^{2\lambda\{k+s\}} f(s)^2 d\gamma ds - \int_0^1\int_{\gamma_+} e^{2\lambda\{k+s\}} f^2(s)d\gamma ds = -\int_0^1\int_{\gamma_+} e^{2\lambda s}[\{I-P_\gamma\}f(s)]^2 d\gamma ds.$$

By the same procedure, we obtain from (55) and the positivity (52):

$$e^{2\lambda N}||f(N)||^2 + \left(\min\{\frac{\delta_0}{2},\frac{\delta_0}{2M}\}e^{-2\lambda} - C_\nu\lambda\right)\sum_{k=0}^{N-1}\int_0^1 e^{2\lambda\{k+s\}}||f_k(s)||_\nu^2 ds$$

$$\le\ ||f(0)||^2 - (1-\frac{\delta_0}{2})\sum_{k=0}^{N-1}\int_0^1\int_{\gamma_+} e^{2\lambda\{k+s\}}[\{I-P_\gamma\}f(s)]^2 d\gamma ds.$$

For $\frac{\delta_0}{2} < 1$ and $C_\nu \lambda < \min\{\frac{\delta_0}{4}, \frac{\delta_0}{4M}\} e^{-2\lambda}$, we have

$$e^{2\lambda N}||f(N)||^2 \leq ||f(0)||^2.$$

Since for $[N, t]$, we have $||f(t)||^2 \leq ||f(N)||^2$, from (54). We therefore conclude the proposition for $e^{2\lambda(t-N)} \leq 2$. ∎

3.1   Strategy for the Proof of Prop. 11

The rest of this section is devoted entirely to the proof of the crucial Proposition 11. The proof of Proposition 11 is based on a contradiction argument. If Proposition 11 were false, then there are no M exists as in Proposition 11 for every linear Boltzmann solution. Hence, for any $k \geq 1$, there exists a sequence of non-zero solutions $f_k(t, x, v)$ to the linearized Boltzmann equation (23) to satisfy one of the following:

(1) **In the in-flow case:** f_k satisfies (11) and

$$\int_0^1 ||\mathbf{P} f_k(s)||_\nu^2 ds \geq k\{\int_0^1 ||(\mathbf{I} - \mathbf{P}) f_k(s)||_\nu^2 ds + \int_0^1 ||f_k(s)||_\gamma^2 ds\}.$$

Equivalently, in terms of normalization $Z_k(t, x, v) \equiv \frac{f_k(t,x,v)}{\sqrt{\int_0^1 ||\mathbf{P} f_k(s)||_\nu^2 ds}}$, we have

$$\int_0^1 ||\mathbf{P} Z_k(s)||_\nu^2 ds \equiv 1, \tag{60}$$

and

$$\int_0^1 ||(\mathbf{I} - \mathbf{P}) Z_k(s)||_\nu^2 ds + \int_0^1 ||Z_k(s)||_\gamma^2 ds \leq \frac{1}{k}. \tag{61}$$

We also have from $[\partial_t + v \cdot \nabla_x + L] f_k = 0$,

$$[\partial_t + v \cdot \nabla_x + L] Z_k = 0. \tag{62}$$

(2) **In the bounce-back case:** f_k satisfies (12), the mass-energy conservation laws (17) and (18), and

$$\int_0^1 ||\mathbf{P} f_k(s)||^2 ds \geq k \int_0^1 ||(\mathbf{I} - \mathbf{P}) f_k(s)||_\nu^2 ds. \tag{63}$$

Hence, the normalized Z_k satisfies (60), (62), and

$$\int_0^1 ||(\mathbf{I} - \mathbf{P}) Z_k(s)||_\nu^2 ds \leq \frac{1}{k} \tag{64}$$

(3) **In the specular reflection case:** f_k satisfies (14), and the mass and energy conservation laws (17), (18). (If the domain Ω has any axis of rotation symmetry (5) then f_k also satisfies (19)). We note that (63), (60), (62) and (64) are all valid for the normalized Z_k.

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(4) **In the diffusive reflection case:** f_k satisfies (16), and the mass conservation (17), (62), and

$$\int_0^1 ||\mathbf{P}f_k(s)||_\nu^2 ds \geq k\{\int_0^1 ||(\mathbf{I}-\mathbf{P})f_k(s)||_\nu^2 ds + \int_0^1 ||\{I - P_\gamma\}f_k(s)||_{\gamma_+}^2 ds\}. \tag{65}$$

The normalized Z_k satisfies (60) and

$$\int_0^1 ||(\mathbf{I}-\mathbf{P})Z_k(s)||_\nu^2 ds + \int_0^1 ||\{I - P_\gamma\}Z_k(s)||_{\gamma_+}^2 ds \leq \frac{1}{k}. \tag{66}$$

In all four cases, there exists $Z(t,x,v)$ such that

$$Z_k \to Z \text{ weakly in } \int_0^1 ||\cdot||_\nu^2 ds,$$

since $\sup_k \int_0^1 ||Z_k(s)||_\nu^2 ds < +\infty$, and from (61), (64), (66) that

$$\int_0^1 ||(\mathbf{I}-\mathbf{P})Z_k(s)||_\nu^2 ds \to 0. \tag{67}$$

Notice that it is straightforward to verify

$$\mathbf{P}Z_k \to \mathbf{P}Z \text{ weakly in } \int_0^1 ||\cdot||_\nu^2 ds.$$

Therefore $(\mathbf{I}-\mathbf{P})Z_k \to (\mathbf{I}-\mathbf{P})Z$ weakly, and $(\mathbf{I}-\mathbf{P})Z = 0$ from (67) so that

$$Z(t,x,v) = \{a(t,x) + v \cdot b(t,x) + |v|^2 c(t,x)\}\sqrt{\mu}. \tag{68}$$

Note $LZ_k = L(\mathbf{I}-\mathbf{P})Z_k$ and $\int_0^1 ||(\mathbf{I}-\mathbf{P})Z_k(s)||_\nu^2 ds \to 0$ in four cases. Letting $k \to \infty$ in (62), we have, in the sense of distributions,

$$\partial_t Z + v \cdot \nabla_x Z = 0. \tag{69}$$

The main strategy is to show, on the one hand, Z has to be zero from (67) and one of the inherited boundary conditions (11), (12), (14) and (16). On the other hand, Z_k will be shown to converge strongly to Z in $\int_0^1 ||\cdot||^2 ds$, and $\int_0^1 ||Z||^2 ds \neq 0$. This leads to a contradiction.

3.2 The Limit Function $Z(t,x,v)$

Lemma 12 *There exists constants a_0, c_1, c_2, and constant vectors b_0, b_1 and ϖ such that $Z(t,x,v)$ takes the form:*

$$\left(\{\frac{c_0}{2}|x|^2 - b_0 \cdot x + a_0\} + \{-c_0 tx - c_1 x + \varpi \times x + b_0 t + b_1\} \cdot v + \{\frac{c_0 t^2}{2} + c_1 t + c_2\}|v|^2\right)\sqrt{\mu}. \tag{70}$$

Moreover, these constants are finite:

$$|a_0| + |c_0| + |c_1| + |c_2| + |b_0| + |b_1| + |\varpi| < +\infty. \tag{71}$$

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Proof. We first derive deduce (70). Notice that by plugging (68) into (69) and comparing coefficients in front of $\sqrt{\mu}, v\sqrt{\mu}, |v|^2\sqrt{\mu}$, we deduce the macroscopic equations with $b = (b^1, b^2, b^3)$:

$$
\begin{aligned}
\partial_{x_i} c &= 0, \text{ for } i = 1, 2, 3 & (72)\\
\partial_t c + \partial_{x_i} b^i &= 0, \text{ for } i = 1, 2, 3 & (73)\\
\partial_{x_j} b^i + \partial_{x_i} b^j &= 0, \text{ for } i \neq j, & (74)\\
\partial_{x_i} a + \partial_t b^i &= 0, \text{ for } i = 1, 2, 3 & (75)\\
\partial_t a &= 0. & (76)
\end{aligned}
$$

Since Ω is simply connected, from (72), $c(t, x) \equiv c(t)$. Similarly, from (73),

$$
\begin{aligned}
b^1(t, x) &= -\partial_t c(t) x_1 + \tilde{b}^1(t, x_2, x_3),\\
b^2(t, x) &= -\partial_t c(t) x_2 + \tilde{b}^2(t, x_1, x_3),\\
b^3(t, x) &= -\partial_t c(t) x_3 + \tilde{b}^3(t, x_1, x_2).
\end{aligned}
$$

To determine $\tilde{b}^1$, we first make use of (74) to get

$$
\partial_{x_2} \tilde{b}^1(t, x_2, x_3) + \partial_{x_1} \tilde{b}^2(t, x_1, x_3) = 0
$$

so that $\partial_{x_2}^2 \tilde{b}^1(t, x_2, x_3) = 0$. Therefore $\tilde{b}^1$ is linear with respect to x_2, and

$$
\tilde{b}^1(t, x_2, x_3) = j^1(t, x_3) x_2 + g^1(t, x_3). \tag{77}
$$

Similarly, we also have $\partial_{x_1}^2 \tilde{b}^2(t, x_1, x_3) = 0$ so that

$$
\tilde{b}^2(t, x_1, x_3) = -j^1(t, x_3) x_1 + g^2(t, x_3). \tag{78}
$$

Next, we make use of another equation of (74) to get

$$
\partial_{x_3} \tilde{b}^2(t, x_1, x_3) + \partial_{x_2} \tilde{b}^3(t, x_1, x_2) = 0
$$

with $\partial_{x_3} \tilde{b}^2(t, x_1, x_3) = -\partial_{x_3} j^1(t, x_3) x_1 + \partial_{x_3} g^2(t, x_3)$. From $\partial_{x_2}^2 \tilde{b}^3(t, x_1, x_2) = 0$, we have

$$
\tilde{b}^3 = j^3(t, x_1) x_2 + g^3(t, x_1) \tag{79}
$$

so that $-\partial_{x_3} j^1(t, x_3) x_1 + \partial_{x_3} g^2(x_3) + j^3(t, x_1) x_2 = 0$. Taking one more x_3 derivative, we get

$$
-\partial_{x_3}^2 j^1(t, x_3) x_1 + \partial_{x_3}^2 g^2(t, x_3) = 0
$$

so that $\partial_{x_3}^2 j^1(t, x_3) = \partial_{x_3}^2 g^2(t, x_3) \equiv 0$. Furthermore, taking two more x_1 derivatives, we have $\partial_{x_1}^2 j^3(t, x_1) = 0$. Hence j^1 and g^2 can be expressed as

$$
\begin{aligned}
j^1(t, x_3) &= l^1(t) x_3 + h^1(t),\\
j^3(t, x_1) &= l^1(t) x_1 - h^2(t),\\
g^2(t, x_3) &= h^2(t) x_3 + m^2(t).
\end{aligned}
$$

Plugging back into (77), (78) and (79), we deduce

$$
\begin{aligned}
\tilde{b}^1 &= (l^1(t)x_3 + h^1(t))x_2 + g^1(t, x_3), \\
\tilde{b}^2 &= -(l^1(t)x_3 + h^1(t))x_1 + h^2(t)x_3 + m^2(t), \\
\tilde{b}^3 &= l^1(t)x_1x_2 - h^2(t)x_2 + g^3(t, x_1).
\end{aligned}
\tag{80}
$$

Finally, from the remaining equation in (74),

$$
\partial_{x_3}\tilde{b}^1(t, x_1, x_3) + \partial_{x_1}\tilde{b}^3(t, x_1, x_2) = 0.
$$

By (80), $l^1 x_2 + \partial_{x_3}g^1(t, x_3) + l^1 x_2 + \partial_{x_1}g^3(t, x_1) = 0$. Hence $l^1 \equiv 0$, g^1 is a linear function of x_3 and g^3 is a linear function of x_1:

$$
\begin{aligned}
g^3(t, x_1) &= h^3(t)x_1 + m^3(t) \\
g^1(t, x_3) &= -h^3(t)x_3 + m^1(t).
\end{aligned}
$$

Therefore, letting $\varpi(t) = -[h^2(t), h^3(t), h^1(t)]$ and $m(t) = [m^1(t), m^2(t), m^3(t)]$, we deduce from a direct computation that

$$
b = -c'(t)x + \varpi(t) \times x + m(t).
$$

We also have $\partial_t^2 b(t, x) \equiv 0$ from (75) and (76). Hence $\partial_t^3 c(t) = 0$ from (73), and $c'(t) = c_0 t + c_1$ so that

$$
c = \frac{c_0 t^2}{2} + c_1 t + c_2.
$$

Hence $\varpi''(t) \times x + m''(t) \equiv 0$ and $\varpi''(t) = m''(t) \equiv 0$. We can denote

$$
b = -\{c_0 t + c_1\}x + \{\varpi'(0)t + \varpi\} \times x + b_0 t + b_1.
$$

where ϖ is a constant vector. Moreover, from (75), $\nabla \times \partial_t b \equiv 0$ so that

$$
\nabla \times \{-c_0 x + \varpi'(0) \times x\} \equiv 0.
$$

This implies that $\varpi'(0) = 0$ and from (75) again,

$$
a = \frac{c_0 |x|^2}{2} - b_0 \cdot x + a_0.
$$

Lastly, to prove (71), we note that for $1 \le i, j \le 3$, functions

$$
|x|^2\sqrt{\mu}, \; x_i\sqrt{\mu}, \; \sqrt{\mu}, \; tx \cdot v\sqrt{\mu}, \; x \cdot v\sqrt{\mu}, \; x \times v\sqrt{\mu}, \; tv\sqrt{\mu}, \; v\sqrt{\mu}, \; t^2|v|^2\sqrt{\mu}, \; t|v|^2\sqrt{\mu}, \; |v|^2\sqrt{\mu}
$$

are linearly independent. Therefore, their coefficients $c_0, c_1, c_2, a_0, b_0, b_1, \varpi$ are bounded by $C\left\{\int_0^1 ||Z(s)||^2 ds\right\}^{1/2}$, which is finite. ∎

3.3 Interior Compactness

Lemma 13 *For any smooth function $\chi(t,x)$ such that $\sup \chi \subset\subset (0,1)\times\Omega$, then up to a subsequence, $\lim_{k\to\infty}\int_0^1 ||\chi\{Z_k - Z\}(s)||^2 ds = 0$.*

Proof. We multiply the equation (62) by χ to get

$$[\partial_t + v\cdot\nabla_x]\{\chi Z_k\} = \{[\partial_t + v\cdot\nabla_x]\chi\}Z_k - \chi L Z_k.$$

Since $\int_0^1 ||Z_k(s)||^2 ds$ is uniformly bounded for the hard potentials, by (60) and (67), we deduce from the Averaging Lemma [DL], $\int \chi(t,x)Z_k(v)\chi_v(v)dv$ are compact in $L^2([0,1]\times\Omega)$ for any smooth cutoff function $\chi_v(v)$ (see [G1]). It then follows that

$$\int \chi Z_k(v)[1,v,|v|^2]\sqrt{\mu}dv$$

are compact in $L^2([0,1]\times\Omega)$. Therefore, up to a subsequence, the macroscopic parts of Z_k satisfy $\chi\mathbf{P}Z_k \to \chi\mathbf{P}Z = \chi Z$ strongly in $L^2([0,1]\times\Omega\times\mathbf{R}^3)$. Therefore, in light of $\int_0^1 ||(\mathbf{I}-\mathbf{P})Z_k(s)||_\nu^2 ds \to 0$ in (67) for all four boundary conditions, the remaining microscopic parts χZ_k satisfy $\lim_{k\to\infty}\int_0^1 ||\chi\{\mathbf{I}-\mathbf{P}\}Z_k(s)||^2 ds = 0$, and our lemma follows. ∎

3.4 No Time Concentration

We first establish L^∞ in time estimate for Z_k to rule out possible concentration in time, near either $t=0$ or $t=1$.

Lemma 14 $\sup_{0\le t\le 1, k\ge 1}||Z_k(t)|| < \infty$.

Proof. Since $\int_0^1 ||f_k(s)||_\gamma^2 < \infty$, $\int_0^1 ||Z_k(s)||_\gamma^2 < \infty$. Therefore, by the standard L^2 estimate for (62), we obtain for $0\le t\le 1$:

$$||Z_k(t)||^2 + \int_0^t ||Z_k(s)||_{\gamma_+}^2 ds + 2\int_0^t (LZ_k, Z_k)(s)ds$$
$$= ||Z_k(0)||^2 + \int_0^t ||Z_k(s)||_{\gamma_-}^2 ds. \tag{81}$$

We first derive an upper bound for $Z_k(t)$. In the case of in flow case (11), because of (61), (60) and $L \ge 0$, we deduce

$$||Z_k(t)||^2 \le ||Z_k(0)||^2 + \frac{1}{k}. \tag{82}$$

Note $\int_0^t ||Z_k(s)||_{\gamma_+}^2 ds = \int_0^t ||Z_k(s)||_{\gamma_-}^2 ds$ for either bounce-back or specular reflection (12) and (14), hence (82) is clearly valid. In the case of diffuse reflection (16), we deduce (82) because

$$\int_0^t ||Z_k(s)||_{\gamma_-}^2 ds = \int_0^t ||P_\gamma Z_k(s)||_{\gamma_+}^2 ds \le \int_0^t ||Z_k(s)||_{\gamma_+}^2 ds.$$

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Next, we derive an upper bound for $Z_k(0)$. We note that

$$\int_0^1 (LZ_k(t), Z_k(t))dt \leq C \int_0^1 ||\{\mathbf{I} - \mathbf{P}\}Z_k||_\nu^2 dt \leq \frac{C}{k}.$$

In the case of the in-flow case (11), by (61) and (81),

$$
\begin{aligned}
||Z_k(t)||^2 &\geq ||Z_k(0)||^2 - \int_0^1 ||Z_k(s)||_{\gamma_+}^2 ds - \int_0^1 (LZ_k, Z_k)(s)ds \\
&\geq ||Z_k(0)||^2 - \frac{C}{k}.
\end{aligned}
\tag{83}
$$

Note that $\int_0^t ||Z_k(s)||_{\gamma_+}^2 ds = \int_0^t ||Z_k(s)||_{\gamma_-}^2 ds$ for either bounce-back and specular reflection (12) or (14), so that (83) is clearly valid. In the case of diffuse reflection (16), (83) is valid because of (66):

$$\int_0^t ||Z_k(s)||_{\gamma_-}^2 ds - \int_0^t ||Z_k(s)||_{\gamma_+}^2 ds = -\int_0^t ||\{I - P_\gamma\}Z_k(s)||_{\gamma_+}^2 ds \geq -\frac{1}{k}.$$

Since $\int_0^1 ||Z_k(t)||^2 dt \leq C \int_0^1 ||Z_k(t)||_\nu^2 dt \leq C\{1 + \frac{1}{k}\}$ for hard potentials, integrating (83) over $0 \leq t \leq 1$ yields

$$
\begin{aligned}
||Z_k(0)||^2 &\leq \int_0^1 ||Z_k(t)||^2 dt + \frac{C}{k} \\
&\leq C \int_0^1 ||Z_k(t)||_\nu^2 dt + \frac{C}{k} \\
&\leq C\{1 + \frac{1}{k}\} + \frac{C}{k},
\end{aligned}
$$

by (60) and (67). Our lemma thus follows from (82). ∎

3.5   No Boundary Concentration

The most delicate step is to prove that there is no concentration at the boundary $\partial\Omega$ so that $Z_k \to Z$ strongly in $[0, 1] \times \bar{\Omega} \times \mathbf{R}^3$. Let

$$\Omega_{\varepsilon^4} \equiv \{x \in \Omega : \xi(x) < -\varepsilon^4\}.$$

To this end, we will establish a careful energy estimate in the thin shell-like region near the boundary $[0, 1] \times \{\Omega \setminus \Omega_{\varepsilon^4}\} \times \mathbf{R}^3$.

Recall $n(x) = \frac{\nabla \xi(x)}{|\nabla \xi(x)|} \neq 0$, well-defined and smooth on $\Omega \setminus \Omega_{\varepsilon^4}$ for ε small. For $m > 1/2$, for any (x, v), we define the outward moving (inward moving) indicator function χ_+ (χ_-) as

$$
\begin{aligned}
\chi_+(x, v) &= \mathbf{1}_{\Omega \setminus \Omega_{\varepsilon^4}}(x)\mathbf{1}_{\{|v| \leq \varepsilon^{-m}, n(x) \cdot v > \varepsilon\}}(v) \\
\chi_-(x, v) &= \mathbf{1}_{\Omega \setminus \Omega_{\varepsilon^4}}(x)\mathbf{1}_{\{|v| \leq \varepsilon^{-m}, n(x) \cdot v < -\varepsilon\}}(v).
\end{aligned}
$$

Our main strategy is to show that the moving (non-grazing) part $\chi_{\pm}Z_k$ are controlled by the inner boundary values of Z_k on $\partial\Omega_{\varepsilon^4} = \{\xi(x) = -\varepsilon^4\}$, which are further controlled by the (compact!) interior parts of Z_k. Hence, no concentration is possible. On the remaining almost grazing part $\{1 - \chi_{\pm}\}Z_k$, thanks to the fact $\int_0^1 ||\{\mathbf{I} - \mathbf{P}\}Z_k(s)||_\nu^2 ds \to 0$, no concentration can occur for the small velocity set $\{|v| \geq \varepsilon^{-m}\} \cup \{|n(x) \cdot v| \leq \varepsilon\}$.

Lemma 15

$$\sup_{k \geq 1} \int_0^1 \int_{\Omega \setminus \Omega_{\varepsilon^4}} \int_{\substack{|n(x)\cdot v| \leq \varepsilon \\ \text{or } |v| \geq \varepsilon^{-m}}} |Z_k(s, x, v)|^2 dx dv ds \leq C\varepsilon. \tag{84}$$

Proof. Let $\mathbf{P}Z_k = \{a_k(t,x) + v \cdot b_k(t,x) + |v|^2 c_k(t,x)\}\sqrt{\mu}$. Since $\sup_k \int_0^1 ||Z_k(s)||^2 ds$ is finite and $[1, v, |v|^2]\sqrt{\mu}$ are linearly independent, there is $C > 0$ (independent of k) such that

$$\int_0^1 ||a_k(s)||^2 ds + \int_0^1 ||b_k(s)||^2 ds + \int_0^1 ||c_k(s)||^2 ds \leq C \int_0^1 ||Z_k(s)||^2 ds \leq C. \tag{85}$$

By (67), we can split:

$$\int_0^1 \int_{\Omega \setminus \Omega_{\varepsilon^4}} \int_{\substack{|n(x)\cdot v| \leq \varepsilon \\ \text{or } |v| \geq \varepsilon^{-m}}} |Z_k(s, x, v)|^2 dx dv ds$$

$$\leq \int_{\substack{|n(x)\cdot v| \leq \varepsilon \\ \text{or } |v| \geq \varepsilon^{-m}}} |\mathbf{P}Z_k(s, x, v)|^2 + \int_{\substack{|n(x)\cdot v| \leq \varepsilon \\ \text{or } |v| \geq \varepsilon^{-m}}} |\{\mathbf{I} - \mathbf{P}\}Z_k(s, x, v)|^2$$

$$\leq \int_0^1 \int_{\Omega \setminus \Omega_{\varepsilon^4}} \int_{\substack{|n(x)\cdot v| \leq \varepsilon \\ \text{or } |v| \geq \varepsilon^{-m}}} |\mathbf{P}Z_k(s, x, v)|^2 dx dv ds + \frac{C}{k}$$

Even with an extra weight $\{1 + |v|^2\}^l$ $(l \geq 0)$, the first term can be bounded by the Fubini Theorem as

$$\int_{\substack{|n(x)\cdot v| \leq \varepsilon \\ \text{or } |v| \geq \varepsilon^{-m}}} \{1 + |v|^2\}^l |\mathbf{P}Z_k(s, x, v)|^2 dx dv ds$$

$$\leq \int_0^1 \int_{\Omega \setminus \Omega_{\varepsilon^4}} \{|a_k^2(s, x)| + |b_k^2(s, x)| + |c_k^2(s, x)|\} \times$$

$$\times \{\int_{\substack{|n(x)\cdot v| \leq \varepsilon \\ \text{or } |v| \geq \varepsilon^{-m}}} \{1 + |v|^2\}^{l+2} \mu dv\} dx ds. \tag{86}$$

We note that the inner $v-$integral above is bounded, uniformly in x. In fact, by a change of variable $v_{||} = \{n(x) \cdot v\}n(x)$, and $v_\perp = v - v_{||}$ for $|n(x) \cdot v| \leq \varepsilon$, the

inner integral is bounded by the sum of

$$\int_{|n(x)\cdot v|\leq\varepsilon}\{1+|v|^2\}^{l+2}\mu dv \;\;\leq\;\; C\int_{-\varepsilon}^{\varepsilon}dv_{||}\int_{\mathbf{R}^2}e^{-|v_\perp|^2/8}dv_\perp \leq C\varepsilon \quad (87)$$

$$\text{and}\quad \int_{|v|\geq\varepsilon^{-m}}\{1+|v|^2\}^{l+2}\mu dv \;\;\leq\;\; C\varepsilon.$$

Our lemma thus follows from (85). ∎

To study the non-grazing parts $\chi_{\pm}Z_k$, we fix $(x,v)\in\{\Omega\setminus\Omega_{\varepsilon^4}\}\times\mathbf{R}^3$ and any moment s such that $\varepsilon\leq s\leq 1-\varepsilon$, and . We define for backward in time $0\leq t\leq s$:

$$\tilde{\chi}_+(t,x,v) = \mathbf{1}_{\Omega\setminus\Omega_{\varepsilon^4}}(x-v\{t-s\})\mathbf{1}_{\{|v|\leq\varepsilon^{-m},n(x-v\{t-s\})\cdot v>\varepsilon\}}(v); \quad (88)$$

and for forward in time $0\leq s\leq t$:

$$\tilde{\chi}_-(t,x,v) = \mathbf{1}_{\Omega\setminus\Omega_{\varepsilon^4}}(x-v\{t-s\})\mathbf{1}_{\{|v|\leq\varepsilon^{-m},n(x-v\{t-s\})\cdot v<-\varepsilon\}}(v). \quad (89)$$

Both $\tilde{\chi}_\pm$ solve the transport equations:

$$\partial_t\tilde{\chi}_\pm + v\cdot\nabla_x\tilde{\chi}_\pm = 0, \quad \tilde{\chi}_\pm(s,x,v) = \chi_\pm(x,v). \quad (90)$$

We first prove that

Lemma 16 *(1) For $0\leq s-\varepsilon^2\leq t\leq s$, if $\tilde{\chi}_+(t,x,v)\neq 0$ then $n(x)\cdot v>\frac{\varepsilon}{2}>0$. Moreover, $\tilde{\chi}_+(s-\varepsilon^2,x,v)\equiv 0$, for $x\in\Omega\setminus\Omega_{\varepsilon^4}$.*

(2) For $s\leq t\leq s+\varepsilon^2\leq 1$, if $\tilde{\chi}_-(t,x,v)\neq 0$, then $n(x)\cdot v<-\frac{\varepsilon}{2}<0$. Moreover, $\tilde{\chi}_-(s+\varepsilon^2,x,v)\equiv 0$, for $x\in\Omega\setminus\Omega_{\varepsilon^4}$.

Proof. It suffices to prove (1), the proof for (2) being exactly the same. First of all, by (88), if $\tilde{\chi}_+(t,x,v)\neq 0$, then $x-v(t-s)\in\Omega\setminus\Omega_{\varepsilon^4}$, $n(x-v\{t-s\})\cdot v>\varepsilon$, and $|v|\leq\varepsilon^{-m}$. Hence for $|t-s|\leq\varepsilon^2$, for any $0\leq\theta\leq 1$,

$$|x-\theta v(t-s)-\{x-v(t-s)\}|\leq\varepsilon^{-m}\varepsilon^2\leq\varepsilon, \quad (91)$$

for $2m<1$. Therefore $x-\theta v(t-s)$ is also near $\partial\Omega$ and $|\nabla n(x-\theta v(t-s))|$ is uniformly bounded. Now,

$$\begin{aligned}
n(x)\cdot v &= n(x-v\{t-s\})\cdot v-\{n(x-v\{t-s\})-n(x)\}\cdot v\\
&> \varepsilon-\sup_{0\leq\theta\leq 1}|\nabla n(x-\theta v\{t-s\})|\times|t-s||v|^2.\\
&\geq \varepsilon-C\varepsilon^{-2m+2}\\
&= \varepsilon[1-C\varepsilon^{-2m+1}]\geq\frac{\varepsilon}{2}, \quad (92)
\end{aligned}$$

for $2m<1$. We thus conclude the first assertion.

To prove the second assertion, let $x\in\Omega\setminus\Omega_{\varepsilon^4}$ so that $-\varepsilon^4\leq\xi(x)<0$. If $\tilde{\chi}_+(s-\varepsilon^2,x,v)>0$, by (88), $-\varepsilon^4\leq\xi(x-\varepsilon^2 v)<0$ and $|v|\leq\varepsilon^{-m}$. But

$$\xi(x-\varepsilon^2 v) = \xi(x)-\varepsilon^2\nabla\xi(x)\cdot v+\varepsilon^2 v\cdot\nabla^2\xi(\bar{x})\cdot\varepsilon^2 v,$$

for some $\bar{x}$ is between x and $x - \varepsilon^2 v$. Since $n(x) = \frac{\nabla \xi(x)}{|\nabla \xi(x)|}$, there exists a constant $C_\xi > 0$ such that

$$-\varepsilon^2 \nabla \xi(x) \cdot v = -\varepsilon^2 |\nabla \xi(x)| n(x) \cdot v < -\frac{\varepsilon^3}{2} |\nabla \xi(x)| < -\frac{C_\xi \varepsilon^3}{2}$$

for $x \in \Omega \setminus \Omega_{\varepsilon^4}$. Here we have used the first assertion that $n(x) \cdot v \geq \frac{\varepsilon}{2}$. Therefore

$$
\begin{aligned}
\xi(x - \varepsilon^2 v) \quad &< \quad 0 - \frac{C_\xi \varepsilon^3}{2} + \varepsilon^2 v \cdot \nabla^2 \xi(\bar{x}) \cdot \varepsilon^2 v \\
&< \quad -\frac{C_\xi \varepsilon^3}{2} + C|\varepsilon^4 v^2| = -\frac{C_\xi \varepsilon^3}{2} \{1 - C\varepsilon^{1-2m}\} < -\varepsilon^4
\end{aligned}
$$

for $2m < 1$, and small ε. This is a contradiction to $-\varepsilon^4 \leq \xi(x - \varepsilon^2 v) < 0$. ∎

Lemma 17 *We have the strong convergence*

$$\lim_{k \to \infty} \int_0^1 ||Z_k(s) - Z(s)||^2 ds = 0$$

and $\int_0^1 ||Z(s)||_\nu^2 > 0$. Moreover, Z defined in (70) satisfies the corresponding boundary conditions (11) with $g \equiv 0$, (12), (14) and (16)

Proof. By (90), we multiply $\tilde{\chi}_\pm$ with (62) to get

$$[\partial_t + v \cdot \nabla_x]\{\tilde{\chi}_\pm Z_k\} = -\tilde{\chi}_\pm L Z_k. \tag{93}$$

Since $\int_0^1 ||Z_k(t)||_\gamma^2 dt < \infty$, applying the L^2 estimate backward in time over the shell-like region $[s - \varepsilon^2, s] \times \{\Omega \setminus \Omega_{\varepsilon^4}\} \times \mathbf{R}^3$ for outgoing part χ_+, we obtain:

$$
\begin{aligned}
|| \quad &\chi_+ Z_k(s)||^2 + \int_{s-\varepsilon^2}^s ||\tilde{\chi}_+ Z_k(t)||_{\gamma_+}^2 dt - \int_{s-\varepsilon^2}^s ||\tilde{\chi}_+ Z_k(t)||_{\gamma_+^\varepsilon}^2 dt = ||Z_k \tilde{\chi}_+(s - \varepsilon^2)||^2 \\
&+ \int_{s-\varepsilon^2}^s ||\tilde{\chi}_+ Z_k(t)||_{\gamma_-}^2 dt - \int_{s-\varepsilon^2}^s ||\tilde{\chi}_+ Z_k(t)||_{\gamma_-^\varepsilon}^2 dt - \int_{s-\varepsilon^2}^s (\tilde{\chi}_+ L Z_k, Z_k)(t) dt, \tag{94}
\end{aligned}
$$

where at the inner boundary $\gamma^\varepsilon \equiv \{x : \xi(x) = -\varepsilon^4\} \times \mathbf{R}^3$, its normal vector is $n(x) = \frac{\nabla \xi(x)}{|\nabla \xi(x)|}$. We notice that by Lemma 16, $\tilde{\chi}_+ \equiv 0$ at $s - \varepsilon^2$, while $\tilde{\chi}_+ \equiv 0$ on γ_- and γ_-^ε, since $n(x) \cdot v > 0$ for $\tilde{\chi}_+(s, x, v) \neq 0$. From $\int_0^1 ||(\mathbf{I} - \mathbf{P}) Z_k(s)||_\nu^2 ds \leq \frac{1}{k}$, we get for k large,

$$
\begin{aligned}
\int_{s-\varepsilon^2}^s (\tilde{\chi}_+ L Z_k, Z_k)(t) dt \quad &\leq \quad \int_0^1 \int_{\Omega \times \mathbf{R}^3} |L\{\mathbf{I} - \mathbf{P}\} Z_k||Z_k|(t) dx dv dt \\
&\leq \quad C \left\{ \int_0^1 ||\{\mathbf{I} - \mathbf{P}\} Z_k(t)||_\nu^2 \right\}^{1/2} \left\{ \int_0^1 ||Z_k(t)||_\nu^2 \right\}^{1/2} \\
&\leq \quad \frac{C}{\sqrt{k}}. \tag{95}
\end{aligned}
$$

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Therefore we can simplify (94) as

$$||\chi_+ Z_k(s)||^2 + \int_{s-\varepsilon^2}^s ||\tilde{\chi}_+ Z_k(t)||^2_{\gamma_+} dt \le \int_{s-\varepsilon^2}^s ||\tilde{\chi}_+ Z_k(t)||^2_{\gamma_+^\varepsilon} dt + \frac{C}{\sqrt{k}}.$$

Similarly, we use L^2 estimate forward in time $[s, s+\varepsilon^2] \times \{\Omega \setminus \Omega_{\varepsilon^4}\} \times \mathbf{R}^3$ for incoming part χ_- to get

$$|| \quad \tilde{\chi}_- Z_k(s + \varepsilon^2)||^2 + \int_s^{s+\varepsilon^2} ||\tilde{\chi}_- Z_k(t)||^2_{\gamma_+} dt - \int_s^{s+\varepsilon^2} ||\tilde{\chi}_- Z_k(t)||^2_{\gamma_+^\varepsilon} dt = ||\chi_- Z_k(s)||^2$$

$$+ \int_s^{s+\varepsilon^2} ||\tilde{\chi}_- Z_k(t)||^2_{\gamma_-} dt - \int_s^{s+\varepsilon^2} ||\tilde{\chi}_- Z_k(t)||^2_{\gamma_-^\varepsilon} dt - \int_s^{s+\varepsilon^2} (\tilde{\chi}_- L\{\mathbf{I} - \mathbf{P}\} Z_k, Z_k)(t) dt.$$

We notice that $\tilde{\chi}_- \equiv 0$ at $t = s + \varepsilon^2$, while $\tilde{\chi}_- \equiv 0$ on the incoming part γ_+ and γ_+^ε by part (2) of Lemma 16. Therefore we deduce from (95)

$$||\chi_- Z_k(s)||^2 + \int_s^{s+\varepsilon^2} ||\tilde{\chi}_- Z_k(t)||^2_{\gamma_-} dt \le \int_s^{s+\varepsilon^2} ||\tilde{\chi}_- Z_k(t)||^2_{\gamma_-^\varepsilon} dt + \frac{C}{\sqrt{k}}.$$

By Lemma 16, the supports of $\tilde{\chi}_\pm Z_k(t)$ on $\gamma_\pm^\varepsilon$ are contained in $|n(x) \cdot v| \ge \frac{\varepsilon}{2}$. Combining the $\pm$ cases, we are able to estimate $\tilde{\chi}_\pm Z_k$ in terms of the inner boundary contributions as

$$\int_{\Omega \setminus \Omega_{\varepsilon^4}} \int_{\substack{|n(x) \cdot v| > \varepsilon \\ |v| \le \varepsilon^{-m}}} |Z_k(s, x, v)|^2 dx dv$$

$$\le \int_s^{s+\varepsilon^2} ||\tilde{\chi}_- Z_k(t)||^2_{\gamma_-^\varepsilon} dt + \int_{s-\varepsilon^2}^s ||\tilde{\chi}_+ Z_k(t)||^2_{\gamma_+^\varepsilon} dt + \frac{2C}{\sqrt{k}}$$

$$\le \int_{s-\varepsilon^2}^{s+\varepsilon^2} ||\mathbf{1}_{\{|v| \le \varepsilon^{-m}, \text{ and } |n(x) \cdot v| \ge \frac{\varepsilon}{2}\}} Z_k(t)||^2_{\gamma^\varepsilon} dt + \frac{2C}{\sqrt{k}}. \tag{96}$$

Since the outward normal at $x \in \partial \Omega_{\varepsilon^4}$ is $n(x)$, the set $\{(x, v) : x \in \partial \Omega_{\varepsilon^4}, |v \cdot n(x)| \ge \frac{\varepsilon}{2}\}$ is away from the singular set $\gamma_0^\varepsilon = \{(x, v) : x \in \partial \Omega_{\varepsilon^4}, |v \cdot n(x)| = 0\}$. Hence, by (40) in Lemma 6, both the backward or forward trajectories emanating from $\partial \Omega_{\varepsilon^4} \times \{|v| \le \varepsilon^{-m}, |v \cdot n(x)| \ge \frac{\varepsilon}{2}\}$ spend a positive period of time inside $\bar{\Omega}_{\varepsilon^4}$. Since

$$\{\partial_t + v \cdot \nabla_x\} \{\mathbf{1}_{\{|v| \le \varepsilon^{-m}\}} (Z_k - Z)\} = -\mathbf{1}_{\{|v| \le \varepsilon^{-m}\}} L\{\mathbf{I} - \mathbf{P}\} Z_k, \tag{97}$$

we can apply Ukai's trace theorem (Theorem 5.1.1, [U1]) to $\mathbf{1}_{\{|v| \le \varepsilon^{-m}\}} (Z_k - Z)$

over $\bar{\Omega}_{\varepsilon^4}$ to get

$$\int_{s-\varepsilon^2}^{s+\varepsilon^2} ||\mathbf{1}_{\{|v|\leq\varepsilon^{-m},|v\cdot n(x)|\geq\frac{\varepsilon}{2}\}}\{Z_k(t) - Z(t)\}||_{\gamma^\varepsilon}^2 ds$$

$$= \int_{s-\varepsilon^2}^{s+\varepsilon^2} ||\mathbf{1}_{\{|v\cdot n(x)|\geq\frac{\varepsilon}{2}\}}\{\mathbf{1}_{\{|v|\leq\varepsilon^{-m}\}}(Z_k(t) - Z(t))\}||_{\gamma^\varepsilon}^2 ds$$

$$\leq C_\varepsilon \int_\varepsilon^{1-\varepsilon} \left\{ ||\mathbf{1}_{\{|v|\leq\varepsilon^{-m}\}}(Z_k(t) - Z(t))||_{\Omega_{\varepsilon^4}\times\mathbf{R}^3}^2 + ||\mathbf{1}_{\{|v|\leq\varepsilon^{-m}\}}\{L\{\mathbf{I}-\mathbf{P}\}Z_k(t)\}||_{\Omega_{\varepsilon^4}\times\mathbf{R}^3}^2 \right\} dt$$

$$\leq C_\varepsilon \int_\varepsilon^{1-\varepsilon} ||Z_k(t) - Z(t)||_{\Omega_{\varepsilon^4}\times\mathbf{R}^3}^2 dt + C_\varepsilon \int_0^1 ||\{\mathbf{I}-\mathbf{P}\}Z_k(t)||_\nu^2 dt$$

$$\leq C_\varepsilon \int_\varepsilon^{1-\varepsilon} ||Z_k(t) - Z(t)||_{\Omega_{\varepsilon^4}\times\mathbf{R}^3}^2 dt + \frac{C_\varepsilon}{k}.$$

Therefore, for fixed ε, we have from the interior compactness in Lemma 13

$$\lim_{k\to\infty} \int_{s-\varepsilon^2}^{s+\varepsilon^2} ||\mathbf{1}_{\{|v|\leq\varepsilon^{-m},|v\cdot n(x)|\geq\varepsilon\}}\{Z_k(t) - Z(t)\}||_{\gamma^\varepsilon}^2 dt = 0.$$

Hence, for k large, and for any $\varepsilon \leq s \leq 1 - \varepsilon$, by (96)

$$\int_{\Omega\backslash\Omega_{\varepsilon^4}} \int_{\substack{|v|\leq\varepsilon^{-m}, \\ |v\cdot n(x)|\geq\varepsilon}} |Z_k(s,x,v)|^2 dxdv$$

$$\leq 2\int_{s-\varepsilon^2}^{s+\varepsilon^2} ||\mathbf{1}_{\substack{|v|\leq\varepsilon^{-m} \\ |v\cdot n(x)|\geq\varepsilon}}\{Z_k(t) - Z(t)\}||_{\gamma^\varepsilon}^2 dt + 2\int_{s-\varepsilon^2}^{s+\varepsilon^2} ||\mathbf{1}_{\substack{|v|\leq\varepsilon^{-m} \\ |v\cdot n(x)|\geq\varepsilon}} Z(t)||_{\gamma^\varepsilon}^2 dt + \frac{2C}{\sqrt{k}}$$

$$\leq \varepsilon + \int_{s-\varepsilon^2}^{s+\varepsilon^2} ||Z(t)||_{\gamma^\varepsilon}^2 ds.$$

But from Lemma 12, $Z(s,x,v)$ is smooth so its trace is given by (70) as well. By (71), since the time interval is small,

$$\int_{s-\varepsilon^2}^{s+\varepsilon^2} ||Z(t)||_{\gamma^\varepsilon}^2 dt \leq 2\varepsilon^2 \times \sup_{0\leq t\leq 1} ||Z(t)||_{\gamma^\varepsilon}^2 \leq C\varepsilon^2,$$

where C depends on $a_0, c_0, c_1, c_2, b_0, b_1$ and ϖ. We thus deduce that for $\varepsilon \leq s \leq 1 - \varepsilon$, for k large,

$$\int_{\Omega\backslash\Omega_{\varepsilon^4}} \int_{\substack{|n\cdot v|\geq\varepsilon \\ |v|\leq\varepsilon^{-m}}} |Z_k(s,x,v)|^2 dxdv \leq C\varepsilon. \tag{98}$$

We are now ready to prove compactness of Z_k. We split

$$\int_0^1 \int \int_\Omega |Z_k(s,x,v) - Z(s,x,v)|^2 dsdxdv = \int_0^\varepsilon + \int_\varepsilon^{1-\varepsilon} + \int_{1-\varepsilon}^1.$$

By Lemma 14, we conclude that the integrals $\int_0^\varepsilon + \int_{1-\varepsilon}^1$ are bounded by $C\varepsilon$. On the other hand, we further split the main part $\int_\varepsilon^{1-\varepsilon}$ as

$$2\int_\varepsilon^{1-\varepsilon}\int_{\Omega\setminus\Omega_{\varepsilon^4}}\int_{\substack{|n\cdot v|\ge\varepsilon\\|v|\le\varepsilon^{-m}}}|Z_k(s,x,v)|^2 + 2\int_\varepsilon^{1-\varepsilon}\int_{\Omega\setminus\Omega_{\varepsilon^4}}\int_{\substack{|n\cdot v|\le\varepsilon\\ \text{or } |v|\ge\varepsilon^{-m}}}|Z_k(s,x,v)|^2$$

$$+2\int_\varepsilon^{1-\varepsilon}\int_{\Omega\setminus\Omega_{\varepsilon^4}}\int|Z(s,x,v)|^2 + \int_\varepsilon^{1-\varepsilon}\int\int_{\Omega_{\varepsilon^4}}|Z_k(s,x,v)-Z(s,x,v)|^2.$$

Clearly, the first term is bounded by (98), the second term is bounded by $C\varepsilon$ thanks to Lemma 15; by (71), the third term is bounded by

$$\int_{\Omega\setminus\Omega_{\varepsilon^4}}\int|Z(t,x,v)|^2 dxdv \le C|\Omega\setminus\Omega_{\varepsilon^4}| \le C\varepsilon,$$

where C depends on $a_0, c_0, c_1, c_2, b_0, b_1$ and ϖ. The last term goes to zero as $k\to\infty$ by Lemma 13. We hence deduce the strong convergence

$$\int_0^1\int\int_\Omega |Z_k(s,x,v)-Z(s,x,v)|^2 dsdxdv \to 0$$

by first letting ε small, then letting $k\to\infty$. From our normalization $\int_0^1\|\mathbf{P}Z_k(s)\|_\nu^2 ds \equiv 1$ with $\mathbf{P}Z_k = \{a_k + v\cdot b_k + c_k|v|^2\}\sqrt\mu$, there exists $C>0$ independent of k such that

$$\int_0^1\|\mathbf{P}Z_k(s)\|^2 ds \ge C\int_0^1\|\mathbf{P}Z_k(s)\|_\nu^2 ds \ge C > 0,$$

because both norms are equivalent to

$$\int_0^1|a_k(s,x)|^2 dxds + \int_0^1|b_k(s,x)|^2 dxds + \int_0^1|c_k(s,x)|^2 dxds.$$

Hence $\int_0^1\|Z(s)\|^2 ds = \lim_{k\to\infty}\int_0^1\|Z_k(s)\|^2 ds \ge C > 0$.

Finally, we study the boundary conditions which Z satisfies. In fact, recalling (97) and $\int_0^1\|Z_k(t)-Z(t)\|^2 dt \to 0$, we use Ukai's trace theorem to conclude, for any fixed $\varepsilon > 0$,

$$\lim_{k\to\infty}\int_0^1\|\mathbf{1}_{\{|v\cdot n(x)|\ge\frac{\varepsilon}{2},|v|\le\frac{1}{\varepsilon^m}\}}Z_k(s)-Z(s)\|_\gamma^2 ds$$

$$\le\ C\lim_{k\to\infty}[\int_0^1\|\mathbf{1}_{|v|\le\frac{1}{\varepsilon^m}}\{Z_k(s)-Z(s)\}\|^2 ds + \int_0^1\|[\partial_t + v\cdot\nabla_x]\mathbf{1}_{|v|\le\frac{1}{\varepsilon^m}}\{Z_k(s)-Z(s)\}\|^2 ds]$$

$$\le\ C\lim_{k\to\infty}\left(\int_0^1\|\mathbf{1}_{|v|\le\frac{1}{\varepsilon^m}}\{L\{\mathbf{I}-\mathbf{P}\}Z_k(t)\}\|^2 dt\right) = 0. \tag{99}$$

For the in-flow boundary case, by (61) and the continuity of Z,

$$\int_0^1\|\mathbf{1}_{\{|v\cdot n(x)|\ge\frac{\varepsilon}{2}\}}\mathbf{1}_{|v|\le\varepsilon^{-m}}Z(s)\|_\gamma^2 ds = \lim_{k\to\infty}\int_0^1\|\mathbf{1}_{\{|v\cdot n(x)|\ge\frac{\varepsilon}{2}\}}\mathbf{1}_{|v|\le\varepsilon^{-m}}Z_k(s)\|_\gamma^2 ds = 0,$$

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so that $Z \equiv 0$ on γ.

For the bounce-back and specular reflections, $Z_k(t,x,v) = Z_k(t,x,-v)$, or $Z_k(t,x,v) = Z_k(t,x,R(x)v)$. Letting $k \to \infty$, we deduce that Z satisfies the same relation respectively for $\{|v \cdot n(x)| \geq \frac{\varepsilon}{2}\}$. Therefore, $Z(t,x,v) = Z(t,x,-v)$ or $Z(t,x,v) = Z(t,x,R(x)v)$ respectively by the continuity of Z.

For the diffusive reflection, notice that on γ_-,

$$Z_k(t,x,v) = c_\mu \{ \int_{n \cdot v' > 0} Z_k(t,x,v') \sqrt{\mu} n \cdot v' dv' \} \sqrt{\mu(v)} \equiv \bar{a}_k(t,x) \sqrt{c_\mu \mu(v)} \quad (100)$$

Fix $\varepsilon > 0$ small and for any $x \in \partial \Omega$, on the set $\{v \cdot n(x) > \varepsilon\}$, $Z_k \to Z$ in $L^2([0,1] \times \gamma)$. This implies that from (99)

$$\left\{ \int_0^1 \int_{\partial \Omega} |\bar{a}_k(t,x)|^2 \int_{\substack{v \cdot n(x) > \varepsilon \\ \text{and } |v| \geq \varepsilon^{-m}}} c_\mu \mu dv \right\} dx dt = \int_0^1 ||\mathbf{1}_{\substack{|v \cdot n(x)| \geq \varepsilon \\ |v| \geq \varepsilon^{-m}}} \{Z_k(t)\}_{\gamma_-} ||^2 dt < C_\varepsilon < \infty.$$

Notice that for ε small, $\int_{\substack{v \cdot n(x) > \varepsilon \\ \text{and } |v| \geq \varepsilon^{-m}}} \mu dv$ is a finite non-zero constant, independent of x. It follows that

$$\left\{ \int_0^1 \int_{\partial \Omega} |\bar{a}_k(t,x)|^2 \right\} dx dt \leq C_\varepsilon < \infty.$$

This implies that $P_\gamma \{Z_k\}_{\gamma_+} \equiv \bar{a}_k(t,x) \sqrt{c_\mu \mu(v)}$ are uniformly bounded in $L^2([0,1] \times \gamma_+)$. But from (66), $\{I - P_\gamma\}\{Z_k\}_{\gamma_+} \to 0$ in $L^2([0,1] \times \gamma_+)$, we deduce that $\{Z_k\}_{\gamma_+}$ are uniformly bounded in $L^2([0,1] \times \gamma_+)$ with a weak limit. But $\{Z_k\}_{\gamma_+} \to Z$ strongly in $L^2([0,1] \times \{\gamma_+ \setminus \gamma_0\})$ by the trace theorem, so that $\{Z_k\}_{\gamma_+} \to Z$ weakly in $L^2([0,1] \times \gamma_+)$ since γ_0 has zero measure. Hence

$$c_\mu \{ \int_{n \cdot v' > 0} Z_k(t,x,v') \sqrt{\mu} n \cdot v' dv' \} \sqrt{\mu(v)} \to c_\mu \{ \int_{n \cdot v' > 0} Z(t,x,v') \sqrt{\mu} n \cdot v' dv' \} \sqrt{\mu(v)}$$

weakly $L^2([0,1] \times \gamma_+)$. We then recover (16) by letting $k \to \infty$ in (100). ∎

3.6 Boundary Condition Leads to $Z = 0$.

Since Z now satisfies one of the boundary conditions $Z_\gamma = 0$, (12), (14), and (16), we will show that Z in (70) has to be zero and this leads to a contradiction.

In the case of in-flow boundary (11), since $Z = 0$ on γ, from (70), for any t and $x \in \partial \Omega$, and $v \in \mathbf{R}^3$, by comparing the coefficients in front of the polynomials of v, we deduce that $\{\frac{c_0 t^2}{2} + c_1 t + c_2\} \equiv 0$ and

$$\{-c_0 tx - c_1 x + \varpi \times x + b_0 t + b_1\} \equiv \{\frac{c_0}{2}|x|^2 - b_0 \cdot x + a_0\} \equiv 0.$$

Therefore $c_0 = c_1 = c_2 = 0$, and $b_0 = 0$. Then $a_0 = 0$ and $\varpi \times x + b_1 \equiv 0$, or

$$\varpi^2 x_3 - \varpi^3 x_2 + b_1^1 = -\varpi^1 x_3 + \varpi^3 x_1 + b_1^2 = \varpi^1 x_2 - \varpi^2 x_1 + b_1^3 \equiv 0 \quad (101)$$

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for all $x \in \partial\Omega$. Notice that since $\xi(x) = 0$ is two dimensional, so we may assume that (x_1, x_2) are (locally) independent. Hence $\varpi^1 = \varpi^2 = b_1^3 = 0$ then $\varpi^3 = b_1^2 = 0$, and finally $b_1^1 = 0$. Therefore we deduce $Z \equiv 0$.

In the case of the bounce-back case (12), for any fixed t, because of (97), we apply Ukai's trace theorem over $[0, t] \times \Omega \times \mathbf{R}^3$ to get, for any $0 \leq t \leq 1$,

$$\lim_{k \to \infty} ||\mathbf{1}_{\{|v| \leq \varepsilon^{-1}\}} \{Z_k(t) - Z(t)\}|| = 0. \tag{102}$$

Therefore, by (17) and (18),

$$\int Z(t)\sqrt{\mu} = \lim_{k \to \infty} \int_{|v| \leq \frac{1}{\varepsilon}} Z_k(t)\sqrt{\mu} + \lim_{k \to \infty} \int_{|v| \geq \frac{1}{\varepsilon}} Z_k(t)\sqrt{\mu} \equiv 0, \tag{103}$$

$$\int Z(t)|v|^2\sqrt{\mu} \equiv \lim_{k \to \infty} \int_{|v| \leq \frac{1}{\varepsilon}} |v|^2 Z_k(t)\sqrt{\mu} + \lim_{k \to \infty} \int_{|v| \geq \frac{1}{\varepsilon}} |v|^2 Z_k(t)\sqrt{\mu}, \tag{104}$$

because the integrations over $|v| \geq \frac{1}{\varepsilon}$ are bounded by $C||Z_k(t)|| \int_{|v| \geq \frac{1}{\varepsilon}} \mu^{1/4} = C\varepsilon$, from the L^∞ estimates in Lemma 14. We therefore obtain that for all t,

$$\int \{\frac{c_0}{2}|x|^2 - b_0 \cdot x + a_0\}\sqrt{\mu} + \{\frac{c_0 t^2}{2} + c_1 t + c_2\}|v|^2\sqrt{\mu} \equiv 0, \tag{105}$$

$$\int \{\frac{c_0}{2}|x|^2 - b_0 \cdot x + a_0\}|v|^2\sqrt{\mu} + \{\frac{c_0 t^2}{2} + c_1 t + c_2\}|v|^4\sqrt{\mu} \equiv 0. \tag{106}$$

This implies that $c_0 = c_1 = 0$. Moreover, since from the bounce-back boundary condition $Z(t, x, v) = Z(t, x, -v)$, we must have $b(t, x) \equiv 0$ in (24), or

$$b(t, x) \equiv \varpi_2 \times x + b_0 t + b_1 \equiv 0$$

for all $x \in \partial\Omega$. Clearly $b_0 = 0$ as a function of t. From the argument after (101), $\varpi_2 = 0 = b_1$. We therefore deduce that from (105) and (106) that

$$\int a_0 \sqrt{\mu} dv + c_2 \int |v|^2 \sqrt{\mu} dv \equiv \int a_0 |v|^2 \sqrt{\mu} + c_2 \int |v|^4 \sqrt{\mu} dv \equiv 0.$$

We thus have $a_0 = c_2 = 0$, then $Z \equiv 0$ for the bounce-back case.

The specular reflection is more delicate. Using the same mass and conservation laws (105) and (106), we again have $c_1 = c_0 = 0$ and $b(t, x) = \varpi_2 \times x + b_0 t + b_1$. Now from the specular reflection, we have for any $x \in \partial\Omega$, $b(t, x) \cdot n \equiv 0$ or

$$\{\varpi \times x + b_0 t + b_1\} \cdot n(x) \equiv 0.$$

Hence $b_0 = 0$ for all $x \in \partial\Omega$ and

$$\{\varpi \times x\} \cdot n(x) + b_1 \cdot n(x) = 0. \tag{107}$$

In the case $\varpi = 0$, we have $b_1 \cdot n(x) \equiv 0$ on $\partial\Omega$. We can choose $x' \in \partial\Omega$ such that $b_1 || n(x')$ by taking the minimizer of $\min_{\xi(x)=0} b_1 \cdot x$. Hence, $b_1 \cdot n(x') = 0$ and $b_1 = 0$.

For $\varpi \neq 0$, let's decompose $b_1 = \beta_1 \frac{\varpi}{|\varpi|} + \beta_2 \eta$, where $|\eta| = 1$ and $\eta \perp \varpi$. Then $\eta = \{\frac{\varpi}{|\varpi|} \times \eta\} \times \frac{\varpi}{|\varpi|}$. Hence

$$b_1 = \beta_1 \frac{\varpi}{|\varpi|} + \beta_2 \eta = \beta_1 \frac{\varpi}{|\varpi|} + \beta_2 \{\frac{\varpi}{|\varpi|^2} \times \eta\} \times \varpi \equiv \beta_1 \frac{\varpi}{|\varpi|} - x_0 \times \varpi.$$

where $x_0 = -\frac{\beta_2}{|\varpi|^2} \varpi \times \eta$. By plugging this back into (107), we get

$$\beta_1 \frac{\varpi}{|\varpi|} \cdot n(x) + \varpi \times (x - x_0) \cdot n(x) = 0.$$

Once again, we can choose a point $x' \in \partial\Omega$ such that $\varpi \parallel n(x')$ (e.g., look for minimizer of $\min_{\xi(x)=0} \varpi \cdot x$). We then deduce $\varpi \times (x' - x_0) \cdot n(x') = 0$ and hence $\beta_1 = 0$. So

$$Z = \varpi \times (x - x_0) \cdot v \sqrt{\mu} \tag{108}$$

and $\varpi \times (x - x_0) \cdot n(x) \equiv 0$ for all $x \in \partial\Omega$. If Ω is not rotational symmetric, there is no non-zero ϖ and x_0 exist, then we deduce that $Z \equiv 0$ for the specular case.

On the other hand, if Ω is rotational symmetric, there are such non-zero ϖ and x_0 for Ω so that (108) is valid, we then have to use the additional conservation law for angular momentum:

$$\int \varpi \times (x - x_0) \cdot v Z(t) \sqrt{\mu} dv = 0$$

as $k \to \infty$ of the same expression for Z_k (see the proof for (103)). Therefore, we combine (108) to get

$$\int \{\varpi \times (x - x_0) \cdot v\}^2 \mu dx dv \equiv 0$$

Therefore $\varpi \times (x - x_0) \cdot v \equiv 0$ and $Z \equiv 0$ from (108).

In the case of diffuse boundary condition (16), because of (17), we have (105) and $c_1 = c_0 = 0$. Moreover, we have

$$Z(t, x, v) = c_\mu \{\int_{n \cdot v' > 0} Z(t, x, v') \sqrt{\mu} n \cdot v' dv'\} \sqrt{\mu}.$$

on γ_-. Since $v\sqrt{\mu}, \sqrt{\mu}, |v|^2 \sqrt{\mu}$ are linearly independent, this implies for all t and $x \in \partial\Omega$,

$$b(t, x) \equiv \varpi \times x + b_0 t + b_1 \equiv 0, \quad \text{and} \quad c_2 \equiv 0.$$

Therefore, $b_0 = 0$, and $\varpi = b_1 = 0$ as in (101). Therefore, we have from (105) $a_0 \int \sqrt{\mu} dv = 0$. Hence $Z \equiv 0$.

4 L^∞ Decay Theory

4.1 L^∞ Decay For In-flow Boundary Condition

4.1.1 G(t,0) and Continuity

As outlined in Section 1.6, we study the L^∞ (pointwise) decay for the weighted $h = wf$ of the linear Boltzmann equation (27) with the in-flow boundary condition. We first derive explicit formula for solution operator $G(t, 0)$ for the homogenous transport equation (29) with in-flow boundary condition. Note that for non-zero in-flow datum at the boundary, $G(t, 0)$ in general is not a semigroup.

Lemma 18 *Let $h_0(x, v) \in L^\infty$ and $wg \in L^\infty$. Let $\{G(t, 0)h_0\}$ be the solution to the transport equation (29)*

$$\{\partial_t + v \cdot \nabla_x + \nu\}G(t, 0)h_0 = 0, \qquad G(0, 0)h_0 = h_0, \quad \{G(t, 0)h_0\}_{\gamma_-} = wg.$$

For any (x, v), with $x \in \bar{\Omega}$, let $t_{\mathbf{b}}(x, v)$ be its back-time exit time defined in Definition 7. Then for a.e. (x, v),

$$
\begin{aligned}
\{G(t, 0)h_0\}(t, x, v) &= \mathbf{1}_{t - t_{\mathbf{b}} \leq 0} e^{-\nu(v)t} h_0(x - tv, v) \\
&\quad + \mathbf{1}_{t - t_{\mathbf{b}} > 0} e^{-\nu(v)t_{\mathbf{b}}} \{wg\}(t - t_{\mathbf{b}}, x - t_{\mathbf{b}}v, v). \quad (109)
\end{aligned}
$$

Moreover,

$$\sup_{t \geq 0} e^{\nu_0 t} ||G(t, 0)h_0||_\infty \leq ||h_0||_\infty + \sup_{s \geq 0} e^{\nu_0 s} ||wg(s)||_\infty. \qquad (110)$$

Proof. For almost every x, v, along the characteristic line $\frac{dx}{d\tau} = v, \frac{dv}{d\tau} = 0$,

$$\frac{d}{d\tau}\{e^{\nu(v)\tau} G(\tau, 0)h_0\} \equiv 0.$$

Hence $e^{\nu(v)\tau} G(\tau, 0)h_0$ is constant along the characteristic. Choose any point (t, x, v) in $[0, \infty) \times \Omega \times \mathbf{R}^3$ with its backward exit point $(t - t_{\mathbf{b}}, x_{\mathbf{b}}, v)$. If $t - t_{\mathbf{b}}(x, v) \leq 0$, then the backward trajectory first hits on the initial plane $\{t = 0\}$. On the other hand, if $t - t_{\mathbf{b}}(x, v) > 0$, then the backward trajectory first hits the boundary. Since $\{G(\tau, 0)h_0\}_{\gamma_-} = wg$ a.e., from part (4) of Lemma 6, (109) is clearly valid for almost every $x, v, x \in \bar{\Omega}$, and estimate (110) follows immediately from (109) with $t_{\mathbf{b}} = t - (t - t_{\mathbf{b}})$. ∎

Lemma 19 *Let Ω be convex as in (4). Let $h_0(x, v)$ be continuous in $\bar{\Omega} \times \mathbf{R}^3 \backslash \gamma_0$, g be continuous in $[0, \infty) \times \{\partial\Omega \times \mathbf{R}^3 \backslash \gamma_0\}$, $q(t, x, v)$ be continuous in the interior of $[0, \infty) \times \Omega \times \mathbf{R}^3$ and $\sup_{[0, \infty) \times \Omega \times \mathbf{R}^3} |\frac{q(t, x, v)}{\nu(v)}| < \infty$. Let $h(t, x, v)$ be the solution of*

$$\{\partial_t + v \cdot \nabla_x + \nu\}h = q, \qquad h(0) = h_0, \qquad h_{\gamma_-} = wg.$$

Assume the compatibility condition on γ_-,

$$h_0(x, v) = \{wg\}(0, x, v) \tag{111}$$

Then $h(t, x, v)$ is continuous on $[0, \infty) \times \{\bar{\Omega} \times \mathbf{R}^3 \setminus \gamma_0\}$.

Proof. Let $(x, v) \notin \gamma_0$ and denote its backward exit time $[t - t_\mathbf{b}, x_\mathbf{b}, v]$. Since $\frac{d}{d\tau}\{e^{\nu(v)\tau}G(\tau, s)h\} = q$ along the characteristic $\frac{dx}{dt} = v$, $\frac{dv}{dt} = 0$, for $t - t_\mathbf{b} \le 0$,

$$h(t, x, v) = e^{-\nu(v)t}h_0(x - vt, v) + \int_0^t e^{-\nu(v)(t-s)}q(s, x - v(t - s), v)ds. \tag{112}$$

If $t - t_\mathbf{b} > 0$, we then have

$$h(t, x, v) = e^{-\nu(v)t_\mathbf{b}}\{wg\}(t - t_\mathbf{b}, x_\mathbf{b}, v) + \int_{t-t_\mathbf{b}}^t e^{-\nu(v)(t-s)}q(s, x - v(t - s), v)ds. \tag{113}$$

Since $(x, v) \notin \gamma_0$, if $x \notin \partial\Omega$, then $\xi(x) < 0$; and if $x \in \partial\Omega$, then $v \cdot \nabla \xi(x) \ne 0$. This implies in (36), $\alpha(t) > 0$. Since ξ is convex and $\xi(x_\mathbf{b}) = 0$, we now apply Velocity Lemma 5 to get

$$\alpha(t - t_\mathbf{b}) = \{v \cdot \nabla\xi(x_\mathbf{b})\}^2 \ge c\alpha(t) > 0. \tag{114}$$

We thus conclude $v \cdot n(x_\mathbf{b}) \ne 0$ and also $t_\mathbf{b}(x, v) > 0$ by (40). Therefore, by Lemma 6, $t_\mathbf{b}(x, v)$, $x_\mathbf{b}(x, v)$ are both smooth functions of (x, v).

Now take any point $(\bar{t}, \bar{x}, \bar{v})$ close to (t, x, v) and we separate three cases. If $t - t_\mathbf{b}(t, x, v) > 0$, when $(\bar{t}, \bar{x}, \bar{v})$ is close to (t, x, v), $\bar{t} - t_\mathbf{b}(\bar{x}, \bar{v}) > 0$ by continuity. Therefore

$$h(\bar{t}, \bar{x}, \bar{v}) = e^{-\nu(\bar{v})\bar{t}_\mathbf{b}}\{wg\}(\bar{t} - \bar{t}_\mathbf{b}, \bar{x}_\mathbf{b}, v) + \int_{\bar{t}-\bar{t}_\mathbf{b}}^{\bar{t}} e^{-\nu(\bar{v})(\bar{t}-s)}q(s, \bar{x} - \bar{v}(\bar{t} - s), \bar{v})ds. \tag{115}$$

From the continuity of g away from γ_0, the second term above tends to the second term in (113). We split the third term into

$$\int_{\bar{t}-\bar{t}_\mathbf{b}}^{\bar{t}} = \int_{\bar{t}-\varepsilon}^{\bar{t}} + \int_{\bar{t}-\bar{t}_\mathbf{b}+\varepsilon}^{\bar{t}-\varepsilon} + \int_{\bar{t}-\bar{t}_\mathbf{b}}^{\bar{t}-\bar{t}_\mathbf{b}+\varepsilon},$$

where $\varepsilon > 0$ is small. The first and the third parts above are small since $\frac{q}{\nu}$ is bounded, from our assumption. Notice that $x - v(t - s)$ is inside the interior of Ω for $\bar{t} - \bar{t}_\mathbf{b} + \varepsilon \le s \le \bar{t} - \varepsilon$, the middle term above tends to $\int_{t-t_\mathbf{b}+\varepsilon}^{t-\varepsilon} e^{-\nu(v)(t-s)}q(s, x - v(t - s), v)ds$ in (113), from the interior continuity of q. Clearly $|h(t, x, v) - h(\bar{t}, \bar{x}, \bar{v})| \to 0$ as $(\bar{t}, \bar{x}, \bar{v}) \to (t, x, v)$ in this case.

In the case $t - t_\mathbf{b}(x, v) < 0$, $x - vt \notin \partial\Omega$. Then for $(\bar{t}, \bar{x}, \bar{v})$ close to (t, x, v), we have $\bar{t} - t_\mathbf{b}(\bar{x}, \bar{v}) < 0$, $\bar{x} - \bar{v}\bar{t} \notin \partial\Omega$, and

$$h(\bar{t}, \bar{x}, \bar{v}) = e^{-\nu(\bar{v})\bar{t}}h_0(\bar{x} - \bar{v}\bar{t}, \bar{v}) + \int_0^{\bar{t}} e^{-\nu(\bar{v})(\bar{t}-s)}q(s, \bar{x} - \bar{v}(t - s), \bar{v})ds. \tag{116}$$

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Since h_0 is continuous away from γ_0, and q is continuous and $\frac{q}{\nu}$ is bounded in the interior, we again deduce that $h(\bar{t}, \bar{x}, \bar{v}) \to h(t, x, v)$ by the same argument as in the first case $t - t_{\mathbf{b}} > 0$.

Lastly, if $t - t_{\mathbf{b}}(x, v) = 0$ and (112) is valid. By (114), $x_{\mathbf{b}} = x - t_{\mathbf{b}}v = x - tv$, and $(x - tv, v) \notin \gamma_0$. Then for any $(\bar{t}, \bar{x}, \bar{v})$ near (t, x, v), $\bar{t} - \bar{t}_{\mathbf{b}}$ could be either > 0 or ≤ 0. If $\bar{t} - \bar{t}_{\mathbf{b}} \leq 0$, then $h(\bar{t}, \bar{x}, \bar{v})$ still has the same expression (116) as $h(t, x, v)$ and $h(\bar{t}, \bar{x}, \bar{v}) \to h(t, x, v)$ as before. On the other hand, if $\bar{t} - \bar{t}_{\mathbf{b}} > 0$, $h(\bar{t}, \bar{x}, \bar{v})$ is given by (115). By the Velocity Lemma 5 and Lemma 6, we have that $|\bar{t} - \bar{t}_{\mathbf{b}}| + |\bar{x}_{\mathbf{b}} - x_{\mathbf{b}}| \to 0$, so that by the previous argument,

$$\lim_{(\bar{t}, \bar{x}, \bar{v}) \to (t, x, v)} h(\bar{t}, \bar{x}, \bar{v}) = e^{-\nu(v)t}\{wg\}(0, x_{\mathbf{b}}, v) + \int_0^t e^{-\nu(v)t} q(s, x - v(t - s), v) ds.$$

But $\{wg\}(0, x_{\mathbf{b}}, v) = h_0(x_{\mathbf{b}}, v)$ by the compatibility condition (111), hence this limit equals to $h(t, x, v)$ given by (112). ∎

4.1.2 Decay of In-flow $U(t, 0)$

Theorem 20 *Let $\{U(t, 0)h_0\}$ be the solution to the weighted linear Boltzmann equation (27) as*

$$\{\partial_t + v \cdot \nabla_x + \nu - K_w\}U(t, 0)h_0 = 0, \quad U(0, 0)h_0 = h_0, \quad \{U(t, 0)h_0\}_{\gamma_-} = wg.$$

There exists $0 < \lambda < \lambda_0$ such that

$$\sup_{t \geq 0} e^{\lambda t}||U(t, 0)h_0||_\infty \leq C\{||h_0||_\infty + \sup_{0 \leq s \leq \infty} e^{\lambda_0 s}||wg(s)||_\infty\}.$$

Proof. By (112) and (113), we have $\{U(t, 0)h_0\}(t, x, v) =$

$$\mathbf{1}_{t - t_{\mathbf{b}} \leq 0} e^{-\nu(v)t} h_0(x - vt, v) + \mathbf{1}_{t - t_{\mathbf{b}} > 0} e^{-\nu(v)t_{\mathbf{b}}}\{wg\}(t - t_{\mathbf{b}}, x_{\mathbf{b}}, v)$$

$$+ \int_{\max\{0, t - t_{\mathbf{b}}\}}^t e^{-\nu(v)(t - s_1)}\{K_w U(s_1, 0)h_0\}(s_1, x - v(t - s_1), v) ds_1.$$

Let $x_1 = x - v(t - s_1)$, $t'_{\mathbf{b}}$ be the exit time for (x_1, v') and $x'_{\mathbf{b}} = x_1 - v't'_{\mathbf{b}}$. We now further iterate this formula to evaluate $\{K_w U(s_1, 0)h_0\}$ as

$$\int_{\mathbf{R}^3} K_w(v, v')\{U(s_1, 0)h_0\}(s_1, x_1, v') dv' \tag{117}$$

$$= \int_{\mathbf{R}^3} K_w(v, v')\mathbf{1}_{s_1 - t'_{\mathbf{b}} \leq 0} e^{-\nu(v')s_1} h_0(x_1 - v's_1, v') dv'$$

$$+ \int_{\mathbf{R}^3} K_w(v, v')\mathbf{1}_{0 < s_1 - t'_{\mathbf{b}}} e^{-\nu(v')t'_{\mathbf{b}}}\{wg\}(s_1 - t'_{\mathbf{b}}, x'_{\mathbf{b}}, v')\} dv'$$

$$+ \int_{\max\{0, s_1 - t'_{\mathbf{b}}\}}^{s_1} e^{-\nu(v')(s_1 - s)} \int K_w(v, v') K_w(v', v'')\{U(s, 0)h_0\}(s, x_1 - v'(s_1 - s), v'') dv' dv'' ds$$

We note that $||K_w h||_\infty \leq C||h||_\infty$ from (45) in Lemma 7. Clearly, since $\nu(v), \nu(v') \geq \nu_0 > 0$ for hard potentials,

$$e^{-\nu(t - s_1)} e^{-\nu(v')(s_1 - s)} \leq e^{-\nu_0(t - s)}, \quad e^{-\nu(v)(t - s_1)} e^{-\nu(v')t'_{\mathbf{b}}} \leq e^{-\nu_0 t} e^{\nu_0(s_1 - t'_{\mathbf{b}})}.$$

Plugging (117) back into $\{U(t,0)h_0\}(t,x,v)$ yields that all the terms except for the last term in (117) are bounded by ($0 < \lambda < \nu_0$):

$$e^{-\nu_0 t}||h_0||_\infty + e^{-\nu_0 t} \sup_{0\leq s \leq \infty} e^{\lambda s}||wg(s)||_\infty +$$

$$+C_K \int_{\min\{0,t-t_{\mathbf{b}}\}}^t \{e^{-\nu_0 t}||h_0||_\infty + e^{-\nu_0 t} \sup_{0\leq s \leq \infty} e^{\nu_0 s}||wg(s)||_\infty\}ds_1$$

$$\leq \quad C_K\{t+1\}e^{-\nu_0 t}\{||h_0||_\infty + \sup_{0\leq s \leq \infty} e^{\nu_0 s}||wg(s)||_\infty\}. \tag{118}$$

We now concentrate on the last term in (117) and split the velocity-time integration into several regions. We first consider the case $|v| \geq N$.

CASE 1: For $|v| \geq N$. Since from (45) with $\varepsilon = 0$ in Lemma 7,

$$\int\int K_w(v,v')K_w(v',v'')dv'dv'' \leq \frac{C_K}{1+|v|} \leq \frac{C_K}{N},$$

By Lemma 7 again, the double-time integration $\int_{\max\{0,t-t_{\mathbf{b}}\}}^t \int_{\max\{0,s_1-t'_{\mathbf{b}}\}}^{s_1}$ for $|v| \geq N$ is controlled by

$$\frac{C_K}{N} \int_0^t \int_0^{s_1} e^{-\nu_0(t-s)}||U(s,0)h_0||_\infty ds ds_1 \quad \leq \tag{119}$$

$$\frac{C_K e^{-\frac{\nu_0 t}{2}}}{N} \sup_s\{e^{\frac{\nu_0 s}{2}}||U(s,0)h_0||_\infty\} \int_0^t \int_0^{s_1} e^{-\frac{\nu_0(t-s)}{2}}ds ds_1 \quad \leq \quad \frac{C_K e^{-\frac{\nu_0 t}{2}}}{N} \sup_s\{e^{\frac{\nu_0 s}{2}}||U(s,0)h_0||_\infty\},$$

where we have split the exponent as

$$e^{-\nu_0(t-s)} = e^{-\frac{\nu_0 t}{2}}e^{-\frac{\nu_0(t-s)}{2}}e^{\frac{\nu_0 s}{2}}, \tag{120}$$

and used the fact $\int_0^t \int_0^{s_1} e^{-\frac{\nu_0(t-s)}{2}}ds ds_1 < +\infty$ by a direct computation.

CASE 2: For $|v| \leq N$, $|v'| \geq 2N$, or $|v'| \leq 2N$, $|v''| \geq 3N$. Notice that we have either $|v'-v| \geq N$ or $|v'-v''| \geq N$, and either one of the following are valid correspondingly:

$$|K_w(v,v')| \leq e^{-\frac{\varepsilon}{8}N^2}|K_w(v,v')e^{\frac{\varepsilon}{8}|v-v'|^2}|, \qquad |K_w(v',v'')| \leq e^{-\frac{\varepsilon}{8}N^2}|K_w(v',v'')e^{\frac{\varepsilon}{8}|v'-v''|^2}|. \tag{121}$$

From (45) in Lemma 7, both $\int |K_w(v,v')e^{\frac{\varepsilon}{8}|v-v'|^2}|$ and $\int |K_w(v',v'')e^{\frac{\varepsilon}{8}|v'-v''|^2}|$ are still finite. By (120), we use (121) to combine the cases of $|v'-v| \geq N$ or $|v'-v''| \geq N$ as:

$$\int_{\max\{0,t-t_{\mathbf{b}}\}}^t \int_{\max\{0,s_1-t'_{\mathbf{b}}\}}^{s_1} \left\{ \int_{|v|\leq N,|v'|\geq 2N,} + \int_{|v'|\leq 2N,|v''|\geq 3N} \right\}$$

$$\leq \quad C_K \int_0^t \int_0^{s_1} ||U(s,0)h_0||_\infty \left\{ \int_{|v|\leq N,|v'|\geq 2N,} |K_w(v,v')|dv' + \sup_{v'} \int_{|v'|\leq 2N,|v''|\geq 3N} |K_w(v',v'')|dv'' \right\}$$

$$\leq \quad C_{\varepsilon,K} e^{-\frac{\varepsilon}{8}N^2} \int_0^t \int_0^{s_1} e^{-\nu_0(t-s)}||U(s,0)h_0||_\infty ds ds_1$$

$$\leq \quad C_{\varepsilon,K} e^{-\frac{\varepsilon}{8}N^2} e^{-\frac{\nu_0 t}{2}} \sup_{s\geq 0}\{e^{\frac{\nu_0 s}{2}}||U(s,0)h_0||_\infty\}. \tag{122}$$

CASE 3: $s_1 - s \leq \varepsilon$, for $\varepsilon > 0$ small. We bound the last term in (117) by

$$\int_{\min\{0,t-t_{\mathbf{b}}\}}^{t} \int_{s_1-\varepsilon}^{s_1} C_K e^{-\nu_0(t-s)} ||U(s,0)h_0||_\infty ds ds_1$$

$$\leq C_K e^{\frac{-\nu_0 t}{2}} \int_0^t \int_{s_1-\varepsilon}^{s_1} e^{\frac{-\nu_0(t-s)}{2}} \{e^{\frac{\nu_0 s}{2}} ||U(s,0)h_0||_\infty\} ds ds_1$$

$$\leq C_K e^{\frac{-\nu_0 t}{2}} \sup_{s\geq 0}\{e^{\frac{\nu_0 s}{2}} ||U(s,0)h_0||_\infty\} \times \int_0^t \int_{s_1-\varepsilon}^{s_1} e^{\frac{-\nu_0(t-s_1)}{2}} ds ds_1$$

$$\leq C_K e^{\frac{-\nu_0 t}{2}} \sup_{s\geq 0}\{e^{\frac{\nu_0 s}{2}} ||U(s,0)h_0||_\infty\} \times \varepsilon \int_0^t e^{\frac{-\nu_0(t-s_1)}{2}} ds_1$$

$$\leq C_K \varepsilon e^{\frac{-\nu_0 t}{2}} \sup_{s\geq 0}\{e^{\frac{\nu_0 s}{2}} ||U(s,0)h_0||_\infty\}. \tag{123}$$

CASE 4. $s_1 - s \geq \varepsilon$, and $|v| \leq N$, $|v'| \leq 2N, |v''| \leq 3N$. This is the last remaining case because if $|v'| > 2N$, it is included in Case 2; while if $|v''| > 3N$, either $|v'| \leq 2N$ or $|v'| \geq 2N$ are also included in Case 2. We now can bound the integral of the third term in (117) by

$$C \int_{\max\{0,t-t_{\mathbf{b}}\}}^{t} \int_B \int_{\max\{0,s_1-t'_{\mathbf{b}}\}}^{s_1-\varepsilon} e^{-\nu_0(t-s)} |K_w(v,v')K_w(v',v'')\{U(s,0)h_0\}(s,x_1-(s_1-s)v',v'')|$$

where $B = \{|v'| \leq 2N, |v''| \leq 3N\}$. By (44), $K_w(v,v')$ has possible integrable singularity of $\frac{1}{|v-v'|}$, we can choose $K_N(v,v',v'')$ smooth with compact support such that

$$\sup_{|p|\leq 3N} \int_{|v'|\leq 3N} |K_N(p,v') - K_w(p,v')| dv' \leq \frac{1}{N}. \tag{124}$$

Splitting

$$K_w(v,v')K_w(v',v'') = \{K_w(v,v') - K_N(v,v')\}K_w(v',v'')$$
$$+\{K_w(v',v'') - K_N(v',v'')\}K_N(v,v') + K_N(v,v')K_N(v',v''),$$

we can use such an approximation (124) to bound the above s_1, s integration by

$$\frac{Ce^{-\frac{\nu_0 t}{2}}}{N} \sup_s\{e^{\frac{\nu_0}{2}s} ||U(s,0)h_0||_\infty\} \times \left\{ \sup_{|v'|\leq 2N} \int |K_w(v',v'')| dv'' + \sup_{|v|\leq 2N} \int |K_N(v,v')| dv' \right\} \tag{125}$$

$$+C \int_{\max\{0,t-t_{\mathbf{b}}\}}^{t} \int_B \int_{\max\{0,s_1-t'_{\mathbf{b}}\}}^{s_1-\varepsilon} e^{-\nu_0(t-s)} |K_N(v,v')K_N(v',v'')|\{U(s,0)h_0\}(s,x_1-(s_1-s)v',v'')|.$$

Note that $x_1 - (s_1-s)v' \in \Omega$ for either $s_1 - t'_{\mathbf{b}} < 0$, $s \geq 0$, or $0 \leq s_1 - t'_{\mathbf{b}} \leq s$. Split

$$\int_{\max\{0,s_1-t'_{\mathbf{b}}\}}^{s_1-\varepsilon} = \int_0^{s_1-\varepsilon} \{\mathbf{1}_{s_1-t'_{\mathbf{b}}<0} + \mathbf{1}_{0\leq s_1-t'_{\mathbf{b}}\leq s}\}.$$

for the last main term in (125). Since $K_N(v,v')K_N(v',v'')$ is bounded, we first integrate over v' to get

$$C_N \int_{|v'|\leq 2N} \{\mathbf{1}_{s_1-t'_{\mathbf{b}}<\tau}(v') + \mathbf{1}_{0\leq s_1-t'_{\mathbf{b}}\leq s}(v')\}|\{U(s,0)h_0\}(s,x_1-(s_1-s)v',v'')|dv'$$

$$\leq C_N \left\{ \int_{|v'|\leq 2N} \mathbf{1}_\Omega(x_1-(s_1-s)v')|\{U(s,0)h_0\}(s,x_1-(s_1-s)v',v'')|^2 dv' \right\}^{1/2}$$

$$\leq \frac{C_N}{\varepsilon^3} \left\{ \int_\Omega |\{U(s,0)h_0\}(y,v'')|^2 dy \right\}^{1/2}.$$

Here we have made a change of variable $y = x_1 - (s_1-s)v' \in \Omega$, and for $s_1-s\geq \varepsilon$, $\frac{dy}{dv'}\geq \frac{1}{\varepsilon^3}$. Denote $U(s,0)h_0 = wf(s)$ so that f is a L^2 solution to the linear Boltzmann equation (23) with $f(0) = \frac{h_0}{w}$ and $f_{\gamma_-} = g$. We then further control the last term in (125) by:

$$\frac{C_N}{\varepsilon^3} \int_{\max\{0,t-t_{\mathbf{b}}\}}^t \int_0^{s_1-\varepsilon} e^{-\nu_0(t-s)} \int_{|v''|\leq 3N} \left\{ \int_\Omega |\{U(s,0)h_0\}(y,v'')|^2 dy \right\}^{1/2} dv'' ds\, ds_1$$

$$\leq \frac{C_N}{\varepsilon^3} \int_0^t \int_0^{s_1-\varepsilon} e^{-\nu_0(t-s)} \left\{ \int_{|v''|\leq 3N} \int_\Omega |\{U(s,0)h_0\}(y,v'')|^2 dy\, dv'' \right\}^{1/2} ds\, ds_1$$

$$\leq \frac{C_N}{\varepsilon^3} \int_0^t \int_0^{s_1-\varepsilon} e^{-\nu_0(t-s)} \left\{ \int_{|v''|\leq 3N} \int_\Omega |f(s,y,v'')|^2 dy\, dv'' \right\}^{1/2} ds\, ds_1$$

$$\leq \frac{C_N}{\varepsilon^3} e^{-\lambda t} \sup_{s\geq 0}\{e^{\lambda s}||f(s)||\} \int_0^t \int_0^{s_1} e^{-\frac{\nu_0}{2}(t-s)} ds\, ds_1 = \frac{C_N}{\varepsilon^3} e^{-\lambda t} \sup_{s\geq 0}\{e^{\lambda s}||f(s)||\} \quad (126)$$

$$\leq \frac{C_N}{\varepsilon^3} e^{-\lambda t} \left[||f(0)|| + \left\{ \int_0^s e^{2\lambda\theta}||g(\theta)||^2_{\gamma_-} d\theta \right\}^{1/2} \right],$$

where we have used crucially part (1) of Theorem 10 with some $0 < \lambda < \frac{\nu_0}{2}$ in the last line. Note that since $\{1+|v|\}w^{-2} \in L^1(\mathbf{R}^3)$, $||f(0)|| = ||w^{-1}h_0|| \leq C||h_0||_\infty$, and $||g||_{\gamma_-} = ||\frac{w}{w}g||_{\gamma_-} \leq C||wg||_\infty$, and $\int_0^s e^{2\{\lambda-\lambda_0\}\theta} d\theta < \infty$, where λ_0 is in Theorem 1. We can then further bound (126) by

$$\frac{C_{N,\lambda}}{\varepsilon^3} e^{-\lambda t} \left[||h_0||_\infty + \sup_{0\leq s\leq\infty} e^{\lambda_0 s}||wg(s)||_\infty \right].$$

In summary, replacing $\nu_0, \frac{\nu_0}{2}$ by λ and combining (118), (123), (119), (122), (125) and (126), we have established, for any $\varepsilon > 0$ and large $N > 0$,

$$\sup_{t\geq 0}\{e^{\lambda t}||U(t,0)h_0||_\infty\} \leq \{\varepsilon+\frac{C_\varepsilon}{N}\} \sup_{s\geq 0}\{e^{\lambda s}||U(s,0)h_0||_\infty\} + C_K \sup_{0\leq s} e^{2\lambda_0 s}||wg(s)||_\infty + C_{\varepsilon,N}||h_0||_\infty.$$

First choosing ε small, then N sufficiently large so that $\{\varepsilon + \frac{C_\varepsilon}{N}\} < \frac{1}{2}$,

$$\sup_{t\geq 0}\{e^{\lambda t}||U(t,\tau)h||_\infty\} \leq 2C_K \sup_{0\leq s\leq\infty} e^{\lambda_0 s}||wg(s)||_\infty + 2C_{\varepsilon,N}||h_0||_\infty,$$

and we conclude our proof. ∎

4.2 L^∞ Decay for the Bounce-Back Reflection

4.2.1   Bounce-Back Cycles and Continuity of $G(t)$

Definition 21 (*Bounce-Back Cycles*) *Let* $(t, x, v) \notin \gamma_0$. *Let* $(t_0, x_0, v_0) = (t, x, v)$ *and inductively define for* $k \geq 1$:

$$(t_{k+1}, x_{k+1}, v_{k+1}) = (t_k - t_{\mathbf{b}}(x_k, v_k), x_{\mathbf{b}}(x_k, v_k), -v_k).$$

We define the back-time cycles as:

$$X_{\mathbf{cl}}(s; t, x, v) = \sum_k \mathbf{1}_{[t_{k+1}, t_k)}(s)\{x_k + (s - t_k)v_k\}, \quad V_{\mathbf{cl}}(s; t, x, v) = \sum_k \mathbf{1}_{[t_{k+1}, t_k)}(s)v_k. \tag{127}$$

Remark 22 *Clearly, we have* $v_{k+1} \equiv (-1)^{k+1}v$, *for* $k \geq 1$,

$$x_k = \frac{1 - (-1)^k}{2}x_1 + \frac{1 + (-1)^k}{2}x_2, \tag{128}$$

and let $d = t_1 - t_2$, *then for* $k \geq 1$,

$$t_k - t_{k+1} = d \geq t - t_{\mathbf{b}} > 0. \tag{129}$$

We follow the outline in Section 1.6 and first establish an abstract lemma.

Lemma 23 *Let* $\mathcal{M}$ *be an operator on* $L^\infty(\gamma_+) \to L^\infty(\gamma_-)$ *such that* $||\mathcal{M}||_{\mathcal{L}(L^\infty, L^\infty)} = 1$. *Then for any* $\varepsilon > 0$, *there exists* $h(t) \in L^\infty$ *and* $h_\gamma \in L^\infty$ *solving*

$$\{\partial_t + v \cdot \nabla_x + \nu\}h = 0, \quad h_{\gamma_-} = (1 - \varepsilon)Mh_{\gamma_+}, \quad h(0, x, v) = h_0 \in L^\infty.$$

Proof. Fix $\varepsilon > 0$, we construct a solution by the following iteration (with $h^0_{\gamma_+} \equiv 0$) for $k = 0, 1, 2....$

$$\{\partial_t + v \cdot \nabla_x + \nu\}h^{k+1} = 0, \quad h^{k+1}_{\gamma_-} = (1 - \varepsilon)Mh^k_{\gamma_+}, \quad h^{k+1}(0, x, v) = h_0.$$

We now show h^k and h^k_γ is a Cauchy sequence. Taking differences, we get

$$\{\partial_t + v \cdot \nabla_x + \nu\}\{h^{k+1} - h^k\} = 0, \quad h^{k+1}_{\gamma_-} - h^k_{\gamma_-} = (1 - \varepsilon)M\{h^k_{\gamma_+} - h^{k-1}_{\gamma_+}\},$$

with zero initial datum $\{h^{k+1} - h^k\}_{t=0} = 0$. Note that from Lemma 18,

$$\sup_s ||h^{k+1}_{\gamma_+}(s) - h^k_{\gamma_+}(s)||_\infty \leq (1 - \varepsilon)\sup_s ||h^k_{\gamma_+}(s) - h^{k-1}_{\gamma_+}(s)||_\infty.$$

Repeatedly using such inequality for $k = 1, 2, ...$, we obtain

$$\sup_s ||h^{k+1}_{\gamma_+}(s) - h^k_{\gamma_+}(s)||_\infty \leq (1 - \varepsilon)^k \sup_s ||h^1_{\gamma_+}(s) - h^0_{\gamma_+}(s)||_\infty.$$

Hence $\{h^k_{\gamma_+}\}$ is Cauchy in $L^\infty(\mathbf{R} \times \gamma_-)$, and then both $\{h^k_{\gamma_-}\}$ and $\{h^k\}$ are Cauchy respectively by Lemma 18. We deduce our lemma by letting $k \to \infty$. $\blacksquare$

Lemma 24 *Let $h_0 \in L^\infty(\Omega \times \mathbf{R}^3)$. There exists a unique solution $G(t)h_0$ of*

$$\{\partial_t + v \cdot \nabla_x + \nu\}\{G(t)h_0\} = 0, \qquad \{G(0)h_0\} = h_0, \tag{130}$$

with the bounce-back reflection $\{G(t)h_0\}(t,x,v) = \{G(t)h_0\}(t,x,-v)$ for $x \in \partial\Omega$. For almost any $(x,v) \in \bar{\Omega} \times \mathbf{R}^3 \setminus \gamma_0$,

$$\{G(t)h_0\}(t,x,v) = \sum_k \mathbf{1}_{[t_{k+1},t_k)}(0)e^{-\nu(v)t}h_0\left(X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0)\right). \tag{131}$$

Moreover, $e^{\nu_0 t}||G(t)h_0||_\infty \le ||h_0||_\infty$.

Proof. For any $\varepsilon > 0$, by Lemma 23, there exists a solution h^ε of

$$\{\partial_t + v \cdot \nabla_x + \nu\}h^\varepsilon = 0, \quad h^\varepsilon(t,x,v) = (1-\varepsilon)h^\varepsilon(t,x,-v), \quad h^\varepsilon(0,x,v) = h_0.$$

with finite $||h^\varepsilon(t,\cdot)||_\infty$ and $\sup_t ||h^\varepsilon_\gamma(t,\cdot)||_\infty$. Such a solution is necessary unique. This is because we can choose $w^{-2}\{1+|v|\} \in L^1$ so that $f^\varepsilon = \frac{h^\varepsilon}{w} \in L^2$ is a L^2 solution to the same equation in (130) with the same boundary condition, with an additional property $\int_0^t ||f^\varepsilon(s)||_\gamma^2 ds < \infty$. Then uniqueness follows from the energy identity for f^ε.

Given any point $(t,x,v) \notin \gamma^0$ and its back-time cycle $[X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s)]$. We notice $|V_{\mathbf{cl}}(s)| = |v|$, for all s, and $\frac{d}{ds}G(s)h_0 \equiv -\nu G(s)h_0$ along the back-time cycle $[X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s)]$ for $t_{k+1} \le s < t_k$. Together with the boundary condition at $s = t_k$, and part (4) of Lemma 6, we deduce that for almost every (x,v), $e^{\nu(v)t}G(s)h_0$ is constant along its back-time cycle $[X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s)]$ in (127). If $(x,v) \in \bar{\Omega} \times \mathbf{R}^3 \setminus \gamma_0$, then $t_{\mathbf{b}}(x,v) > 0$, and

$$h^\varepsilon(t,x,v) = \sum_k \mathbf{1}_{[t_{k+1},t_k)}(0)[1-\varepsilon]^k e^{-\nu(v)t}h_0\left(X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0)\right),$$

where the summation over k is finite for finite t by (129). For all ε,

$$e^{\nu_0 t}||h^\varepsilon(t)||_\infty \le ||h_0||_\infty, \qquad \sup_{t \ge s, \gamma_-} |h^\varepsilon(t,x,v)| \le \sup_{t \ge s, \gamma_+} |h^\varepsilon(t,x,v)| \le ||h_0||_\infty,$$

uniformly bounded. We thus can construct the solution h to (130) with the original bounce-back boundary condition by taking $w-*$ limit: $h(t,x,v) = \lim_{\varepsilon \to 0} h^\varepsilon(t,x,v)$, and $h_\gamma(t,x,v) = \lim_{\varepsilon \to 0} h^\varepsilon_\gamma(t,x,v)$. We thus deduce our lemma by letting $\varepsilon \to 0$. Once again, such a solution $h(t,x,v)$ is necessarily unique in the L^∞ class because $f_\gamma = \frac{h_\gamma}{w} \in L^2_{loc}(L^2(\gamma))$. $\blacksquare$

Lemma 25 *Let ξ be convex as in (4). Let h_0 be continuous in $\bar{\Omega} \times \mathbf{R}^3 \setminus \gamma_0$ and $q(t,x,v)$ be continuous in the interior of $[0,\infty) \times \Omega \times \mathbf{R}^3$ and $\sup_{[0,\infty) \times \Omega \times \mathbf{R}^3} |\frac{q(t,x,v)}{\nu(v)}| < \infty$. Assume the compatibility condition on γ_- : $h_0(x,v) = h_0(x,-v)$. Then the solution $h(t,x,v)$ of*

$$\{\partial_t + v \cdot \nabla_x + \nu\}h = q, \qquad h(0,x,v) = h_0, \tag{132}$$

with $h(t,x,v) = h(t,x,-v)$, $x \in \partial\Omega$ is continuous on $[0,\infty) \times \{\bar{\Omega} \times \mathbf{R}^3 \setminus \gamma_0\}$.

Proof. Take any point $(t, x, v) \notin [0, \infty) \times \gamma_0$ and denote its backward exit point $[t_{\mathbf{b}}, x_{\mathbf{b}}, v]$ along the trajectory. Recall its back-time cycle and (24). Assume $t_{m+1} \le 0 < t_m$. Since $\frac{d}{d\tau}\{e^{\nu(v)}h\} = q$ along the characteristics, $h(t, x, v)$ takes the form

$$e^{-\nu(v)t}h_0\left(x_m - t_m v_m, v_m\right) + \tag{133}$$

$$\sum_{k=0}^{m-1} \int_{t_{k+1}}^{t_k} e^{-\nu(v)(t-s)} q\left(s, x_k + (s - t_k)v_k, v_k\right) ds + \int_0^{t_m} e^{-\nu(v)(t-s)} q\left(s, x_m + (s - t_m)v_m, v_m\right) ds.$$

Since Ω is convex and $(x, v) \notin \gamma_0$, then from the Velocity Lemma 5, $n(x_1) \cdot v_1 \ne 0$. Notice that $x_k \in \partial\Omega$ and $\xi(x_k) = 0$ for $k \ge 1$ so that

$$\alpha(t_k) = (v_k \cdot \nabla \xi(x_k))^2.$$

We now apply the Velocity Lemma 5 to conclude $\alpha(t_k) > C\alpha(t) > 0$ and $v_k \cdot n(x_k) \ne 0$ for all $k \ge 1$. By Lemma 6, t_k, and x_k for $1 \le k \le n$ are smooth functions of (t, x, v). For any other point $(\bar{t}, \bar{x}, \bar{v})$ which is close to (t, x, v), we deduce that $\bar{t}_m > 0$.

In the case that $t_{m+1} < 0$, or equivalently, $x_m - t_m x_m \in \Omega$, from continuity, $\bar{t}_{m+1}(\bar{t}, \bar{x}, \bar{v}) < 0$. Therefore, $h(\bar{t}, \bar{x}, \bar{v})$ has the same expression as $h(t, x, v)$ in (133). Therefore, $h(\bar{t}, \bar{x}, \bar{v}) \to h(t, x, v)$ because $\bar{x}_k \to x_k$, and $\bar{t}_k \to t_k, \bar{v}_k \to v_k$ for $1 \le k \le m + 1$, as in the proof of Lemma 19.

On the other hand, if $t_{m+1} = 0$, or equivalently, $x_{m+1} = x_m - t_m x_m \in \partial\Omega$, and $(x_{m+1}, v_m) \notin \gamma_0$. Then by continuity, we know that $\bar{t}_{m+1}(\bar{t}, \bar{x}, \bar{v})$ is close to zero. In the case that $\bar{t}_{m+1}(\bar{t}, \bar{x}, \bar{v}) \le 0$, then (133) is again valid and the continuity follows as before. However, if $\bar{t}_{m+1}(\bar{t}, \bar{x}, \bar{v}) > 0$, then $\bar{t}_{m+2}(\bar{t}, \bar{x}, \bar{v}) < 0$ (because $t_{m+2} < 0$), and we have a different expression for $h(\bar{t}, \bar{x}, \bar{v})$ as:

$$e^{-\nu(\bar{v})\bar{t}}h_0\left(\bar{x}_{m+1} - \bar{t}_{m+1}\bar{v}_{m+1}, \bar{v}_{m+1}\right) + \sum_{k=0}^{m} \int_{\bar{t}_{k+1}}^{\bar{t}_k} e^{-\nu(\bar{v})(\bar{t}-s)} q\left(s, \bar{x}_k + (s - \bar{t}_k)\bar{v}_k, \bar{v}_k\right) ds$$

$$+ \int_0^{\bar{t}_{m+1}} e^{-\nu(\bar{v})(\bar{t}-s)} q\left(s, \bar{x}_{m+1} + (s - \bar{t}_{m+1})\bar{v}_{m+1}, \bar{v}_{m+1}\right) ds. \tag{134}$$

The last term in (134) goes to zero because $\bar{t}_{m+1} \to 0$, and the second term in (134) tends to the q integration in (133) since $\int_0^{t_m} = \int_{t_{m+1}}^{t_m}$. We now show that the first term (134) tends to the first term in (133) as well. Since $\bar{x}_{m+1} - \bar{t}_{m+1}\bar{v}_{m+1} \to \bar{x}_{m+1} \to x_{m+1} = x_m - t_m v_m \in \partial\Omega$, the first term in (134) tends to

$$h_0\left(x_m - t_m x_m, v_m\right) = h_0(x_m - t_m x_m, -v_m),$$

which is exactly the first term in (133), from the compatibility condition $h_0(x, v) = h_0(x, -v)$ on γ. We therefore conclude the continuity. ∎

4.2.2 Non-Grazing Condition $|S_x| = 0$.

The following lemma is due to Hongjie Dong:

Lemma 26 *For any* $x \in \bar{\Omega}$, *define the set*

$$S_x(v) = \{v \in \mathbf{R}^3 : v \cdot n(x_{\mathbf{b}}(x,v)) = 0\} = \{v \in \mathbf{R}^3 : v \cdot \nabla \xi(x - t_{\mathbf{b}}(x,v)v) = 0\}. \tag{135}$$

If $\partial\Omega$ *is* C^1, *then* $|S_x(v)| = 0$, *where* $|\cdot|$ *is the Lebesaue measure.*

Proof. We first note that if $v \in S_x(v)$, then $kv \in S_x(v)$ for all $k > 0$. It therefore suffices to show that the surface measure $|S_x(v) \cap \mathbf{S}^2| = 0$.

We fix $x \in \bar{\Omega}$ and recall $x_{\mathbf{b}} = x - t_{\mathbf{b}}v \in \partial\Omega$. If $x \in \partial\Omega$, we require $v \cdot n(x) \neq 0$ (a zero measure set). Hence $\frac{v}{|v|} = -\frac{x_{\mathbf{b}} - x}{|x_{\mathbf{b}} - x|}$ and $x_{\mathbf{b}} \neq x$. Our goal is to show that, if $v \in S_x(v)$, then $\frac{v}{|v|}$ is a critical value of the mapping from $\partial\Omega \to \mathbf{S}^2$:

$$\phi(y) = -\frac{y - x}{|y - x|} \tag{136}$$

at $y = x_{\mathbf{b}}$. Since $\phi(y)$ is smooth for $y \neq x$, by Sard's theorem, $\frac{v}{|v|}$ has zero measure in $\mathbf{S}^2$ and our lemma is valid.

Indeed, we assume locally around $x_{\mathbf{b}}$, $\partial\Omega = (y_1, y_2, \eta(y_1, y_2))$ and if $v \in S_x(v)$, then at $(y_1, y_2) = (x_{\mathbf{b}1}, x_{\mathbf{b}2})$:

$$0 = \frac{v}{|v|} \cdot n(x_{\mathbf{b}}) = \{x_{\mathbf{b}1} - x_1\}\partial_1\eta + \{x_{\mathbf{b}2} - x_2\}\partial_1\eta - \eta(x_{\mathbf{b}1}, x_{\mathbf{b}2}) + x_3 = 0. \tag{137}$$

Clearly, since $x_{\mathbf{b}} \neq x$, from (137), $[x_{\mathbf{b}1} - x_1, x_{\mathbf{b}2} - x_2] \neq 0$. But $[\partial_{y_1}\phi, \partial_{y_2}\phi] =$

$$\begin{bmatrix} -\frac{1}{|y-x|} + \frac{(y_1-x_1)^2+(y_1-x_1)(\eta-x_3)\partial_1\eta}{|y-x|^3} & +\frac{(y_1-x_1)\{y_2-x_2+(\eta-x_3)\partial_2\eta\}}{|y-x|^3} \\ \frac{(y_2-x_2)\{y_1-x_1+(\eta-x_3)\partial_1\eta\}}{|y-x|^3} & -\frac{1}{|y-x|} + \frac{(y_2-x_2)^2+(y_2-x_2)(\eta-x_3)\partial_2\eta}{|y-x|^3} \\ -\frac{\partial_1\eta}{|y-x|} + \frac{(\eta-x_3)\{y_1-x_1+(\eta-x_3)\partial_1\eta\}}{|y-x|^3} & -\frac{\partial_2\eta}{|y-x|} + \frac{(\eta-x_3)\{y_2-x_2+(\eta-x_3)\partial_2\eta\}}{|y-x|^3} \end{bmatrix}, \tag{138}$$

by (137), a direct comupation yields

$$\{x_{\mathbf{b}1} - x_1\}\partial_{y_1}\phi + \{x_{\mathbf{b}2} - x_2\}\partial_{y_2}\phi = 0 \tag{139}$$

at $(y_1, y_2) = (x_{\mathbf{b}1}, x_{\mathbf{b}2})$. This implies that $\frac{v}{|v|}$ is a critical value of ϕ. ∎

Lemma 27 *Assume* $|v| \leq 2N$. *Then for any* $\varepsilon > 0$, *there exist* $\delta_{\varepsilon,N} > 0$, *and* $l_{\varepsilon,N,\xi}$ *balls* $B(x_1; r_1), B(x_2, r_2)..., B(x_l; r_l) \subset \bar{\Omega}$, *as well as open sets* $O_{x_1}, O_{x_2},...O_{x_l}$ *of the velocity* v *with* $|O_{x_i}| < \varepsilon$ *for* $1 \leq i \leq l$, *such that for any* $x \in \bar{\Omega}$, *there exists* x_i *so that* $x \in B(x_i; r_i)$ *and for* $v \notin O_{x_i}$,

$$|v \cdot n(x - t_{\mathbf{b}}(x,v)v)| > \delta_{\varepsilon,N} > 0, \quad |v \cdot n(x + t_{\mathbf{b}}(x, -v)v)| > \delta_{\varepsilon,N} > 0. \tag{140}$$

Proof. Fix $\varepsilon > 0$. For any $x \in \bar{\Omega}$, since $|S_x| = 0$ by Lemma 26, there exists an open set O_x^+ such that $|O_x^+| < \varepsilon/2$, and $|v \cdot n(x - t_{\mathbf{b}}(x,v)v)| \neq 0$, for $v \notin O_x^+$. But from part (2) of Lemma 6, this implies that $v \cdot n(x - t_{\mathbf{b}}(x,v)v)$ is a smooth function on the compact set $\{|v| \leq 2N\} \cap \{O_x^+\}^c$. Hence, there exists $\delta_{x,\varepsilon,N} > 0$, such that on the set $\{|v| \leq 2N\} \cap \{O_x^+\}^c$,

$$|v \cdot n(x - t_{\mathbf{b}}(x,v)v)| \geq \delta_{x,\varepsilon,N} > 0. \tag{141}$$

In particular, for $-v \notin O_x^+$ and $|v| \leq 2N$, or equivalently, $v \notin -O_x^+ \equiv \{-v' : v' \in O_x^+\}$, $|v \cdot n(x + t_{\mathbf{b}}(x, -v)v)| > \delta_{x,\varepsilon,N} > 0$. We define $O_x \equiv O_x^+ \cup \{-O_x^+\}$, clearly $|O_x| < \varepsilon$. But by part (2) of Lemma 6, for such $v \notin O_x$, both $t_{\mathbf{b}}(x, v)$ and $t_{\mathbf{b}}(x, -v)$ are smooth functions of both variables x and v. In other words, there exists $B(x; r_x)$ such that if $y \in B(x; r_x)$ and $v \notin O_x$

$$|v \cdot n(y \mp t_{\mathbf{b}}(y, \pm v)v)| > \delta_{x,\varepsilon,N}/2 > 0.$$

Now for any $x \in \bar{\Omega}$, all $B(x, r_x)$ form an open covering for the compact set $\bar{\Omega}$, hence there is a finite $l-$ subcovering $B(x_1; r_1), B(x_2, r_2)..., B(x_l; r_l)$. From our construction, for any $x \in \bar{\Omega}$, there exists i, so that $x \in B(x_i, r_i)$ and moreover, $|v \cdot \nabla n(x \mp t_{\mathbf{b}}(\pm v)v)| > \delta_{x_i,\varepsilon,N}/2 > 0$. We conclude our lemma by choosing $\delta_{\varepsilon,N} = \min_i \frac{\delta_{x_i,\varepsilon,N}}{2}$. $\blacksquare$

4.2.3 L^∞ Decay of Bounce-back $U(t)$

Theorem 28 *Assume $w^{-2}\{1 + |v|\} \in L^1$. Let $h_0 = wf_0 \in L^\infty$. There exits a unique solution $f(t, x, v)$ to the linear Boltzmann equation (23) satisfying $f(0, x, v) = f_0$, and $h(t, x, v) = U(t)h_0$ to the weighted linear Boltzmann equation (27) satisfying $h(0, x, v) = h_0$, both with the bounce-back boundary condition. Then there exist $\lambda > 0$ and $C > 0$ such that*

$$e^{\lambda t}||U(t)h_0||_\infty \leq C||h_0||_\infty. \tag{142}$$

For the well-posedness for both problems, we know from the Duhamel principle (30), there exists a L^∞ solution $h(t) = U(t)h_0$ to the weighted linear Boltzamnn equation (27). By Ukai's trace theorem, it follows that h_γ is also in L^∞. Therefore, since $w^{-2}\{1+|v|\} \in L^1$, $f = \frac{h}{w} \in L^2$ and $f_\gamma = \frac{h_\gamma}{w} \in L^2_{loc}(L^2(\gamma))$ is a solution to the original linear Boltzmann equation (23), which is unique by the standard energy estimate. Based on the L^2 decay estimate for f, to prove the decay estimate, it suffices to establish a finite-time estimate (143).

Lemma 29 *Assume that there exists $\lambda > 0$ so that the solution $f(t, x, v)$ of (23) satisfies $e^{\lambda t}||f(t)|| \leq C||f_0||$. Let $h_0 = wf_0 \in L^\infty$ and $h(t) = U(t)h_0 = wf(t)$ is the solution of (27) where $w^{-2} \in L^1$. Assume there exist $T_0 > 0$ and $C_{T_0} > 0$ such that the satisfies*

$$||U(T_0)h_0||_\infty \leq e^{-\lambda T_0}||h_0||_\infty + C_{T_0} \int_0^{T_0} ||f(s)||ds. \tag{143}$$

Then (142) is valid.

Proof. It suffices to only prove (142) for $t \geq 1$. For any $m \geq 1$, we apply the finite-time estimate (143) repeatedly to functions $h(lT_0 + s)$ for $l = m - 1, m -$

$2, ... 0$:

$$
\begin{aligned}
||h(mT_0)||_\infty &\leq e^{-\lambda T_0}||h(\{m-1\}T_0)||_\infty + C_{T_0}\int_0^{T_0}||f(\{m-1\}T_0+s)||ds\\
&= e^{-\lambda T_0}||h(\{m-1\}T_0)||_\infty + C_{T_0}\int_{\{m-1\}T_0}^{mT_0}||f(s)||ds\\
&\leq e^{-2\lambda T_0}||h(\{m-2\}T_0)||_\infty + e^{-\lambda T_0}C_{T_0}\int_{\{m-2\}T_0}^{\{m-1\}T_0}||f(s)||ds\\
&\quad + C_{T_0}\int_{\{m-1\}T_0}^{mT_0}||f(s)||ds\\
&\leq e^{-m\lambda T_0}||h(0)||_\infty + C_{T_0}\sum_{k=0}^{m-1}e^{-k\lambda T_0}\int_{\{m-k-1\}T_0}^{\{m-k\}T_0}||f(s)||ds,
\end{aligned}
$$

where $h(t) = U(t)h_0$. Now by the L^2 decay assumption, in the interval $\{m-k-1\}T_0 \leq s \leq \{m-k\}T_0$, we have $||f(s)|| \leq e^{-\lambda s}||f_0|| \leq e^{-\lambda\{m-k-1\}T_0}||f_0||$. Hence, $||h(mT_0)||_\infty$ is further bounded by

$$
\begin{aligned}
&e^{-m\lambda T_0}||h_0||_\infty + C_{T_0}\sum_{k=0}^{m-1}e^{-k\lambda T_0}\int_{\{m-k-1\}T_0}^{\{m-k\}T_0}e^{-\lambda\{m-k-1\}T_0}||f_0||ds\\
&\leq e^{-m\lambda T_0}||h_0||_\infty + C_{T_0}e^{\lambda T_0}mT_0e^{-m\lambda T_0}||f_0||\\
&\leq C_{T_0,\lambda}e^{-\frac{m\lambda T_0}{2}}||h_0||_\infty,
\end{aligned}
$$

where by $w^{-2} \in L^1$, $||f_0|| = ||w^{-1}h_0|| \leq C||h_0||_\infty$ and $mT_0e^{-m\lambda T_0} \leq e^{-\frac{m\lambda T_0}{2}}$. For any t, we can find m such that $mT_0 \leq t \leq \{m+1\}T_0$, and

$$
||h(t)||_\infty \leq C||h(mT_0)||_\infty \leq C_{T_0,\lambda}e^{-\frac{m\lambda T_0}{2}}||h_0||_\infty \leq \{C_{T_0,\lambda}e^{\frac{\lambda T_0}{2}}\}e^{-\frac{\lambda}{2}t}||h_0||_\infty,
$$

since $e^{-\frac{m\lambda T_0}{2}} \leq e^{-\frac{\lambda}{2}t}e^{\frac{\lambda T_0}{2}}$. $\blacksquare$

Proof. of Theorem 28: By Lemma 29, we only need to prove the finite-time estimate (143). We use the double Duhamal Principle (31) for semigroup $U(t)$ and $G(t)$. We first estimate the first term in (31) by Lemma 24,

$$
e^{\nu_0 t}||G(t)h_0||_\infty \leq ||h_0||_\infty. \tag{144}
$$

For the second term in (31), we note that $||K_w h||_\infty \leq C||h||_\infty$, then by Lemma 24,

$$
\left\|\int_0^t G(t-s_1)K_w G(s_1)h_0 ds_1\right\|_\infty \leq \int_0^t e^{-\nu_0(t-s_1)}||K_w G(s_1)h_0||_\infty ds_1 = Cte^{-\nu_0 t}||h_0||_\infty. \tag{145a}
$$

We now concentrate on the third term in (31) with the double time integral. We now fix any point (t, x, v) so that $(x, v) \notin \gamma_0$ with its bounce-back cycle.

Using (131) twice, we obtain

$$
\begin{aligned}
&G(t - s_1)K_w G(s_1 - s)K_w h(s)\\
&= e^{-\nu(v)(t-s_1)}\{K_w G(s_1 - s)K_w h(s)\}\,(s_1, X_{\mathbf{cl}}(s_1), V_{\mathbf{cl}}(s_1))\\
&= e^{-\nu(v)(t-s_1)}\int K_w(V_{\mathbf{cl}}(s_1), v')\{G(s_1 - s)K_w h(s)\}\,(s_1, X_{\mathbf{cl}}(s_1), v')\,dv'\\
&= e^{-\nu(v)(t-s_1)}\int\int K_w(V_{\mathbf{cl}}(s_1), v')K_w(V'_{\mathbf{cl}}(s), v'')e^{-\nu(v')(s_1-s)}h\,(s, X'_{\mathbf{cl}}(s), v'')\,dv'dv''
\end{aligned}
$$

where $X'_{\mathbf{cl}}(s) \equiv X_{\mathbf{cl}}(s; s_1, X_{\mathbf{cl}}(s_1), v')$, and $V'_{\mathbf{cl}}(s) \equiv V_{\mathbf{cl}}(s; s_1, X_{\mathbf{cl}}(s_1), v')$. In the case that $|v| \geq N$, we use the same argument in Case 1, (119) in the proof of Theorem 20 to conclude:

$$
\int_0^t \int_0^{s_1} ||G(t-s_1)K_w G(s_1-s)K_w h(s)||_\infty ds ds_1 \leq \frac{C_K}{N}e^{-\frac{\nu_0 t}{2}}\sup_s\{e^{\frac{\nu_0 s}{2}}||h(s)||_\infty\}.
$$
(146)

Moreover, since $|V_{\mathbf{cl}}(s_1)| = |v|$, $|V'_{\mathbf{cl}}(s_1)| = |v'|$, hence as in Case 2, (122) in the proof of Theorem 20, for $|v| \leq N, |v'| \geq 2N$ or $|v| \leq N, |v'| \leq 2N, |v''| \geq 3N$, we deduce for ε small,

$$
\begin{aligned}
&\int_0^t \int_0^{s_1} \int_{\substack{|v|\leq N, |v'|\geq 2N\\ \text{or } |v|\leq N, |v'|\leq 2N, |v''|\geq 3N,}} e^{-\nu_0(t-s)}K_w(V_{\mathbf{cl}}(s_1), v')K_w(V'_{\mathbf{cl}}(s), v'')h\,(s, X'_{\mathbf{cl}}(s_1), v'')\\
&\leq\ C_{\varepsilon,N}e^{-\frac{\varepsilon}{8}N^2}e^{-\frac{\nu_0 t}{2}}\sup_s\{e^{\frac{\nu_0}{2}s}||h(s)||_\infty\}.
\end{aligned}
$$
(147)

We need to only consider the case $|v| \leq N, |v'| \leq 2N, |v''| \leq 3N$, for which we can use the same approximation in Case 4, (125) to obtain an upper bound

$$
\frac{C}{N}e^{-\frac{\nu_0 t}{2}}\sup_s\{e^{\frac{\nu_0}{2}s}||h(s)||_\infty\}
$$
(148)
$$
+C_N \int_0^t \int_0^{s_1} \int_{|v|\leq N, |v'|\leq 2N, |v''|\leq 3N,} e^{-\nu_0(t-s)}|h\,(s, X'_{\mathbf{cl}}(s), v'')|.
$$

Recall $\Omega_{\varepsilon^4} = \{x : \xi(x) < -\varepsilon^4\}$. We focus on the second main term in (148) and separate two cases:

CASE 1: $X_{\mathbf{cl}}(s_1) \in \bar{\Omega} \setminus \Omega_{\varepsilon^4}$ and $\{v' : |v' \cdot \frac{\nabla\xi(X_{\mathbf{cl}}(s_1))}{|\nabla\xi(X_{\mathbf{cl}}(s_1))|}| \leq \varepsilon\}$. In this case, since $|\nabla\xi(X_{\mathbf{cl}}(s_1))| \neq 0$ for ε small, the second term in (148) is bounded by

$$
\begin{aligned}
&C_N \int_0^t \int_0^{s_1} e^{-\nu_0(t-s)}\int_{\{v':|v'\cdot\frac{\nabla\xi(X_{\mathbf{cl}}(s_1))}{|\nabla\xi(X_{\mathbf{cl}}(s_1))|}|\leq\varepsilon, |v'|+|v''|\leq 5N\}} dv'||h(s)||_\infty\\
&\leq\ C_N \varepsilon e^{-\frac{\nu_0 t}{2}}\sup_s\{e^{\frac{\nu_0 s}{2}}||h(s)||_\infty\}\int_0^t \int_0^{s_1} e^{-\frac{\nu_0(t-s)}{2}}ds ds_1\\
&\leq\ C_N \varepsilon e^{-\frac{\nu_0 t}{2}}\sup_s\{e^{\frac{\nu_0 s}{2}}||h(s)||_\infty\}.
\end{aligned}
$$
(149)

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CASE 2: $|v| \leq N, |v'| \leq 2N, |v''| \leq 3N$ and either $X_{\mathbf{cl}}(s_1) \in \Omega_{\varepsilon^4}$ or $X_{\mathbf{cl}}(s_1) \in \bar{\Omega} \setminus \Omega_{\varepsilon^4}$ but $\{v' : |v' \cdot \frac{\nabla \xi(X_{\mathbf{cl}}(s_1))}{|\nabla \xi(X_{\mathbf{cl}}(s_1))|}| > \varepsilon\}$. We denote such a set of v, v' and v'' by A. By using the formula for cycles (128), we get

$$\int_0^t e^{-\nu(v)(t-s)} \int_A \sum_k \int_{t'_{k+1}}^{t'_k} \mathbf{1}_{[0,s_1]}(s) h\left(s, X_{\mathbf{cl}}(s_1) + \sum_{l=0}^{k-1}(t'_{l+1} - t'_l)(-1)^l v' + (s - t'_k)(-1)^k v', v''\right).$$
(150)

We first claim that the number of bounces are bounded on A:

$$k \leq C_{T_0, N, \varepsilon}.$$
(151)

Proof of the claim (151): In the first case $X_{\mathbf{cl}}(s_1) \in \Omega_{\varepsilon^4}$, we have from the mean-value theorem,

$$0 = \xi(X_{\mathbf{cl}}(s_1) - t'_{\mathbf{b}}(X_{\mathbf{cl}}(s_1), v')v') = \xi(X_{\mathbf{cl}}(s_1)) - t'_{\mathbf{b}}v' \cdot \nabla \xi(\bar{x}).$$

Since $|v'| \leq 2N$, and $|\nabla \xi(\bar{x})| \leq C$,

$$t'_{\mathbf{b}} \geq \frac{|\xi(X_{\mathbf{cl}}(s_1))|}{|v' \cdot \nabla \xi(\bar{x})|} \geq \frac{\varepsilon^4}{C_N}.$$

Because $t'_k - t'_{k+1} \geq t'_{\mathbf{b}}$, $k \leq \frac{C_N T_0}{\varepsilon^4}$ and (151) is valid.

In the case $X_{\mathbf{cl}}(s_1) \in \bar{\Omega} \setminus \Omega_{\varepsilon^4}$, and $\{v' : |v' \cdot \frac{\nabla \xi(X_{\mathbf{cl}}(s_1))}{|\nabla \xi(X_{\mathbf{cl}}(s_1))|}| > \varepsilon\}$, denote $t'_{\mathbf{b}}(v') = t'_{\mathbf{b}}(X_{\mathbf{cl}}(s_1), v')$ and $t'_{\mathbf{b}}(-v') = t'_{\mathbf{b}}(X_{\mathbf{cl}}(s_1), -v')$. We expand

$$0 = \xi(X_{\mathbf{cl}}(s_1) - t'_{\mathbf{b}}(v')v') = \xi(X_{\mathbf{cl}}(s_1)) - t'_{\mathbf{b}}(v')\nabla \xi(X_{\mathbf{cl}}(s_1)) \cdot v' + \frac{\{t'_{\mathbf{b}}(v')\}^2}{2} v' \nabla^2 \xi(x_+)v',$$

$$0 = \xi(X_{\mathbf{cl}}(s_1) + t'_{\mathbf{b}}(-v')v') = \xi(X_{\mathbf{cl}}(s_1)) + t'_{\mathbf{b}}(-v')\nabla \xi(X_{\mathbf{cl}}(s_1)) \cdot v' + \frac{\{t'_{\mathbf{b}}(-v')\}^2}{2} v' \nabla^2 \xi(x_-)v'.$$

Since $\nabla^2 \xi(x_\pm)$ are bounded, $|v'| \leq 2N$, and $-\varepsilon^4 < \xi(X_{\mathbf{cl}}(s_1)) < 0$, for some constant C_N, we have

$$-t'_{\mathbf{b}}(v')\nabla \xi(X_{\mathbf{cl}}(s_1)) \cdot v' + C_N\{t'_{\mathbf{b}}(v')\}^2 > 0,$$
$$t'_{\mathbf{b}}(-v')\nabla \xi(X_{\mathbf{cl}}(s_1)) \cdot v' + C_N\{t'_{\mathbf{b}}(-v')\}^2 > 0.$$

We thus have $t'_{\mathbf{b}}(v') \geq \frac{\nabla \xi(X_{\mathbf{cl}}(s_1)) \cdot v'}{C_N}$ and $t'_{\mathbf{b}}(-v') \geq -\frac{\nabla \xi(X_{\mathbf{cl}}(s_1)) \cdot v'}{C_N}$. Since $|\nabla \xi(X_{\mathbf{cl}}(s_1)) \cdot v'| \geq C_\xi \varepsilon$, either $t'_{\mathbf{b}}(v') \geq C_\xi \varepsilon$ or $t'_{\mathbf{b}}(-v') \geq C_\xi \varepsilon$. But for bounce-back cycles,

$$t'_k - t'_{k+1} = t_1 - t_2 = t'_{\mathbf{b}}(v') + t'_{\mathbf{b}}(-v') \geq C_\xi \varepsilon,$$
(152)

for all $k \geq 1$. We therefore have verified the claim (151).

We are now ready to estimate (150). By Lemma 27, for the given $\varepsilon > 0$, there is $\delta_{\varepsilon, N} > 0$, and $[B(x_i, r_i), O_{x_i}]$ for $i = 1, ..., l$, $|O_{x_i}| < \varepsilon$. For $X_{\mathbf{cl}}(s_1) \in \bar{\Omega}$, there exists i such that $X_{\mathbf{cl}}(s_1) \in B(x_i, r_i)$ and for $v' \notin O_{x_i}$

$$|v' \cdot n(X_{\mathbf{cl}}(s_1) \mp t'_{\mathbf{b}}(\pm v')v')| \geq \delta_{\varepsilon, N} > 0.$$

Hence $t'_1 = s_1 - t_{\mathbf{b}}(x, v')$, $t'_2 = t_1 - t_{\mathbf{b}}(x, v') - t_{\mathbf{b}}(x, -v')$ are both also smooth with bounded derivatives over $(s, v') \in [0, T_0] \times \{O^c_{x_i} \cap |v'| \leq 2N\}$. It thus follows from Lemma 6 that t'_l and x'_l are all smooth functions of $v' \notin O_{x_i}$. We then split $\{v' : |v'| \leq 2N\}$ into

$$\int_A = \int_{A \cap \{v' \in O_{x_i}\}} + \int_{A \cap \{v' \in O^c_{x_i}\}} \tag{153}$$

Since $\sum_k \int_{t'_{k+1}}^{t'_k} = \int_0^{s_1}$, the first part is bounded by

$$\int_0^t e^{-\nu_0(t-s)} \int_{v' \in O_{x_i}, |v''| \leq 3N} \sum_k \int_{t'_{k+1}}^{t'_k} \mathbf{1}_{[0, s_1]}(s) |h\left(s, X'_{\mathbf{cl}}(s), v''\right)| \, dv' dv''$$

$$\leq C_N \sup_s \{e^{\frac{\nu_0}{2}s} \|h(s)\|_\infty\} \int_0^t e^{-\nu_0 t} \int_{v' \in O_{x_i}, |v''| \leq 3N} \sum_k \int_{t'_{k+1}}^{t'_k} e^{\frac{\nu_0}{2}s} ds$$

$$\leq C_N |O_{x_i}| \sup_s \{e^{\frac{\nu_0}{2}s} \|h(s)\|_\infty\} \int_0^t \int_0^{s_1} e^{-\nu_0 t} e^{\frac{\nu_0}{2}s} ds ds_1$$

$$= C_N \varepsilon e^{-\frac{\nu_0}{2}t} \times \sup_s \{e^{\frac{\nu_0}{2}s} \|h(s)\|_\infty\}. \tag{154}$$

By (151), we therefore only need to consider the second part in (153):

$$\int_0^t \int_{v' \notin O_{x_i}, |v'| \leq 2N, |v''| \leq 3N} \sum_k^{C_{T_0, N, \varepsilon}} \int_{t'_{k+1}}^{t'_k} e^{-\nu_0(t-s)} \mathbf{1}_{[0, s_1]}(s) |h\left(s, X'_{\mathbf{cl}}(s), v''\right)| \, ds ds_1 dv' dv''.$$

We now wish to change variables as

$$X'_{\mathbf{cl}}(s) \equiv X_{\mathbf{cl}}(s_1) + \sum_{l=0}^{k-1} (t'_{l+1} - t'_l)(-1)^l v' + (s - t'_k)(-1)^k v' \to y.$$

Since for $1 \leq l \leq k$, t'_l is a smooth function of v' on the set of integration: $\{v' \notin O_{x_i}, |v'| \leq 2N\}$, we can expand the determinant as a cubic function of s:

$$\det \left| \frac{\partial y}{\partial v'} \right| = (-1)^k s^3 + q_1(v') s^2 + q_2(v') s + q_3(v'),$$

where $q_i(v)$ are smooth functions of v'. Therefore, by the analytical formula of the algebraic cubic equation, there exists up to three (real) continuous functions $\eta_j(v')$ for $1 \leq j \leq 3$ so that

$$\left\{ (s, v') : \det \left| \frac{\partial y}{\partial v'} \right| = 0 \right\} = \cup_j \{s : s = \eta_j(v')\}.$$

For $\varepsilon > 0$, we then split

$$\int_0^t \int_{v' \notin O_{x_i}, |v'| \leq 2N, |v''| \leq 3N} \sum_k \int_{t'_{k+1}}^{t'_k} \mathbf{1}_{\cup_j \{|s - \eta_j| \leq \varepsilon\}} + \mathbf{1}_{\cap_i \{|s - \eta_j| \geq \varepsilon\}}.$$

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The first part with a small s interval $\{|s - \eta_j| \leq \varepsilon\}$ is bounded by

$$C \sum_{k,i} \int_0^t \int_{v' \notin O_{x_i}, |v'| \leq 2N, |v''| \leq 3N} \int_{t'_{k+1}}^{t'_k} e^{-\nu(v)(t-s)} ||h(s)||_\infty \mathbf{1}_{\{|s-\eta_j| \leq \varepsilon_1, 0 \leq s \leq s_1\}}(s)$$

$$\leq \quad C \sup_s \{e^{\frac{\nu_0}{2}s} ||h(s)||_\infty\} \int_0^t \int_{|v'| \leq 2N, |v''| \leq 3N} e^{-\nu_0 t} \left\{ \sum_{k,j} \int_{t'_{k+1}}^{t'_k} \mathbf{1}_{\{|s-\eta_j| \leq \varepsilon, 0 \leq s \leq s_1\}}(s) e^{\frac{\nu_0}{2}s} ds \right\} ds_1$$

$$\leq \quad C_N \varepsilon e^{-\frac{\nu_0}{2}t} \sup_s \{e^{\frac{\nu_0}{2}s} ||h(s)||_\infty\}. \tag{155}$$

On the other hand, for the second main term, we notice that on the compact set in $0 \leq s \leq T_0$ and $v' \in \cap_j \{|s - \eta_j| \geq \varepsilon\} \cap \{v' \notin O_{x_i}, |v'| \leq 2N\}$, the function $J\{\frac{\partial y}{\partial v'}\}$ is uniformly continuous, with uniformly bounded derivatives for s, v'. There exists a $\zeta_{\varepsilon, N, T_0} > 0$ such that

$$J\{\frac{\partial y}{\partial v'}\} \geq \zeta_{\varepsilon, N, T_0} > 0. \tag{156}$$

And for any point $(s, v') \in [0, T_0] \times \{\cap_j \{|s - \eta_j| \geq \varepsilon\} \cap \{v' \in O_{x_i}, |v'| \leq 2N\}$, there exists open set $O_{s,v'}$ such that $v' \to y$ is one-to-one and invertible. We therefore have a finite covering (depending on ε, T_0,N) O_{s_m, v'_m} such that

$$\cap_j \{|s - r_j| \geq \varepsilon\} \cap \{v' \in O^c, |v'| \leq 2N\} \subset \cup_m O_{s_m, v'_m}$$

and $v' \to y$ is invertible on each O_{s_m, v'_m}. We therefore can change variable $v' \to y$ locally as

$$\int_0^t e^{-\nu_0(t-s)} \sum_{k,m} \int_{t'_{k+1}}^{t'_k} \int_{O_{s_m,v'_m}} \int_{|v''| \leq 3N} \mathbf{1}_{[0,s_1]}(s) h(s, X'_{\mathbf{cl}}(s), v'') \, dv' dv'' ds ds_1$$

$$\leq \quad C_N \int_0^t e^{-\nu_0 t} \sum_{k,m} \int_{t'_{k+1}(y)}^{t'_k(y)} \int_{O_{s_m,v'_m}} \int_{|v''| \leq 3N} \mathbf{1}_{[0,s_1]}(s) e^{\nu_0 s} h(s, y, v'') \frac{1}{J\{\frac{\partial y}{\partial v'}\}} dy dv'' ds ds_1$$

$$\leq \quad C_{N,\varepsilon,T_0} \int_0^t e^{-\nu_0 t} \int_0^{s_1} \int_{|v''| \leq 3N} e^{\nu_0 s} \left\{ \int_\Omega h^2(s, y, v'') \, dy \right\}^{1/2} dv'' \, ds ds_1$$

$$\leq \quad C_{N,\varepsilon,T_0} \int_0^t e^{-\nu_0 t} \int_0^{s_1} e^{\nu_0 s} \left\{ \int_{\Omega \times |v''| \leq 3N} h^2(s, y, v'') \, dy dv'' \right\}^{1/2} ds ds_1$$

$$\leq \quad C_{N,\varepsilon,T_0} \int_0^t e^{-\nu_0 t} \int_0^{s_1} e^{\nu_0 s} ||f(s)|| ds ds_1 \qquad (f = \frac{h}{w})$$

$$\leq \quad C_{N,\varepsilon,T_0} \times \int_0^{T_0} ||f(s)|| ds,$$

where $k \leq C_{T_0, N, \varepsilon}$ and $m \leq C_{T_0, N, \varepsilon}$. We thus conclude from (155), (154), (149), (148), (147), (146), for $t \leq T_0$, $e^{\frac{\nu_0}{2}t} ||h(t)||_\infty$ is bounded by

$$C_K(1+t) e^{-\frac{\nu_0}{2}t} ||h_0||_\infty + (\frac{C_K}{N} + C_N \varepsilon) \sup_{s \leq t} e^{\frac{\nu_0}{2}s} ||h(s)||_\infty + C_{N,\varepsilon,T_0} \int_0^{T_0} ||f(s)|| ds.$$

We first choose T_0 so that $2C_K(1 + T_0)e^{-\frac{\nu_0}{2}T_0} = e^{-\lambda T_0}$, next choose N large, then ε sufficiently small to get $(\frac{C_K}{N} + C_N\varepsilon) < \frac{1}{2}$. We therefore conclude

$$\sup_{0 \leq t \leq T_0} e^{\frac{\nu_0}{2}t}||h(t)||_\infty \leq 2C_K(1 + T_0)||h_0||_\infty + 2C_{N,\varepsilon,T_0} \int_0^{T_0} ||f(s)||ds.$$

We thus conclude our theorem by letting $t = T_0$ on the left hand side. ∎

4.3 L^∞ Decay for Specular Reflection

4.3.1 Specular Cycles and Continuity of $G(t)$

Definition 30 *Fix any point* $(t, x, v) \notin \gamma_0$, *and define* $(t_0, x_0, v_0) = (t, x, v)$, *and for* $k \geq 1$

$$(t_{k+1}, x_{k+1}, v_{k+1}) = (t_k - t_{\mathbf{b}}(t_k, x_k, v_k), x_{\mathbf{b}}(x_k, v_k), R(x_{k+1})v_k), \qquad (157)$$

where $R(x_{k+1})v_k = v_k - 2(v_k \cdot n(x_{k+1}))n(x_{k+1})$. *And we define the specular back-time cycle*

$$X_{\mathbf{cl}}(s) \equiv \sum_{k=1}^\infty \mathbf{1}_{[t_k, t_{k+1})}(s)\left\{x_k + v_k(s - t_k)\right\}, \quad V_{\mathbf{cl}}(s) \equiv \sum_{k=1}^\infty \mathbf{1}_{[t_k, t_{k+1})}(s)v_k.$$

$$(158)$$

Lemma 31 *Let* Ω *be convex (4). Let* $h_0 \in L^\infty$ *and* $G(t)h_0$ *solves (130) with specular boundary condition* $h(t, x, v) = h(t, x, R(x)v)$ *for* $x \in \partial\Omega$. *Then for almost all* $(x, v) \notin \gamma_0$,

$$\{G(t)h_0\}(t, x, v) = e^{-\nu(v)t}h_0(X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0))$$
$$= \sum_k \mathbf{1}_{[t_{k+1}, t_k)}(0)e^{-\nu(v)t}h_0(x_k - t_k v_k, v_k). \qquad (159)$$

And $e^{\nu_0 t}||G(t)h_0||_\infty \leq ||h_0||_\infty$.

Proof. The existence and uniqueness of the solution follows exactly the argument in the proof in Lemma 24, with the bounce-back condition replaced by the specular reflection.

If $(x, v) \in \bar{\Omega} \times \mathbf{R}^3 \setminus \gamma_0$, then $t_{\mathbf{b}}(x, v) > 0$. We consider the back-time specular cycles of (t, x, v) as $[X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s)]$ as in (157). Clearly, $|V_{\mathbf{cl}}(s)| \equiv v$. Since $\frac{d}{ds}\{e^{-\nu(v)}G(s)h_0\} \equiv 0$ for $t_{k+1} < s < t_k$, any k, by part 4 of Lemma 6 and the specular boundary condition at t_{k+1} and t_k, it follows that $e^{-\nu(v)}G(s)h_0$ is a constant along the cycle $[X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s)]$.

We now show for fixed t, the number of bounces k is finite. Since $(x, v) \in \bar{\Omega} \times \mathbf{R}^3 \setminus \gamma_0$, by (36), $\alpha(t) > 0$. By repeatedly applying Velocity Lemma 5 along the back-time cycle $[X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s)]$, we have for all $k \geq 1 : e^{-Ct_k}\alpha(t_k) \geq e^{-Ct_{k-1}}\alpha(t_{k-1}) \geq ... \geq e^{-Ct}\alpha(t) > 0$. But $\alpha(t_k) = \{v_k \cdot \nabla\xi(x_k)\}^2$, we then have

$$\{v_k \cdot n(x_k)\}^2 \geq C\alpha(t) > 0, \qquad (160)$$

for all $k \geq 1$, where C depends on t and v. Therefore by (40) in Lemma 6, that $t_k - t_{k+1} \geq \frac{\delta(t)}{C(t,v)|v|^2} > 0$. So that the summation over k is finite. ∎

Lemma 32 *Let ξ be convex as in (4). Let h_0 be continuous in $\bar{\Omega} \times \mathbf{R}^3 \setminus \gamma_0$ and $q(t,x,v)$ be continuous in the interior of $[0,\infty) \times \Omega \times \mathbf{R}^3$ and $\sup_{[0,\infty) \times \Omega \times \mathbf{R}^3} |\frac{q(t,x,v)}{\nu(v)}| < \infty$. Assume that on γ_-, $h_0(x,v) = h_0(x,R(x)v)$. Then the specular solution $h(t,x,v)$ to (132) is continuous on $[0,\infty) \times \{\bar{\Omega} \times \mathbf{R}^3 \setminus \gamma_0\}$.*

Proof. We only sketch the proof, which is similar to that for Lemma 25. Take any point $(t,x,v) \notin [0,\infty) \times \gamma_0$ and consider its specular back-time cycle $[X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s)]$ as in (159). By repeatedly applying the Velocity Lemma 5 and Lemma 6, it follows that $t_k(t,x,v), x_k(t,x,v)$ and $v_k(t,x,v)$ are all smooth functions of (t,x,v). We assume that $t_{m+1} \leq 0 < t_m$, then $h(t,x,v)$ is given by (133) with specular cycles $[t_k, x_k, v_k] \in [X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s)]$. For any other point $(\bar{t}, \bar{x}, \bar{v})$ which is close to (t,x,v). We now show that $h(\bar{t}, \bar{x}, \bar{v})$ is close to $h(t,x,v)$ by separating two different cases.

In the case that $t_{m+1} < 0$, or equivalently, $x_m - (t_m - s)v_m \in \Omega$, away from the boundary. By continuity, $\bar{t}_{m+1} < 0$. Therefore, we have the same expression for $h(\bar{t}, \bar{x}, \bar{v})$ as $h(t,x,v)$ in (133) with t_k, x_k, v_k replaced by $\bar{t}_k, \bar{x}_k, \bar{v}_k$. Therefore, since $|v_l| \equiv |v|$, $h(\bar{t}, \bar{x}, \bar{v}) \to h(t,x,v)$ following from the continuity of $\bar{t}_l \to t_l$, $\bar{x}_l \to x_l$, $\bar{v}_l \to v_l$.

On the other hand, in the case $t_{m+1} = 0$, $x_{m+1} = x_m - t_m v_m \in \partial\Omega$. From (160), $(x_{k+1}, v_k) \notin \gamma_0$. Then by continuity, we know that $\bar{t}_m > 0$, and $\bar{t}_{m+1}$ is close to zero. In the case that $\bar{t}_{m+1}(\bar{t}, \bar{x}, \bar{v}) < 0$, then (133) is still valid and the continuity follows. However, if $\bar{t}_{m+1} > 0$, then $\bar{t}_{m+2} < 0$, due to $t_{m+2} < t_{m+1} = 0$. Therefore $h(\bar{t}, \bar{x}, \bar{v})$ is given by a different expression (134) with specular cycles $[t_k, x_k, v_k] \in [X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s)]$. We notice that since $\bar{t}_{m+1} \to 0$, the $q-$integrals in (134) tend to $q-$integrals in (133) because of $\int_0^{t_m} = \int_{t_{m+1}}^{t_m}$. On the other hand, since $\bar{x}_{m+1} - \bar{t}_{m+1} \bar{v}_{m+1} \to x_{m+1}$, $\bar{v}_{m+1} \to v_{m+1} = R(x_m)v_m$, the first term in (134) tends to the first term in (133) as

$$h_0\left(\bar{x}_{m+1} - \bar{t}_{m+1}\bar{v}_{m+1}, \bar{v}_{m+1}\right) \to h_0\left(x_{m+1}, R(x_m)v_m\right) = h_0\left(x_m, v_m\right),$$

from $h_0(x,v) = h_0(x, R(x)v)$ on γ. We thus complete the proof. ∎

4.3.2 $\det\left(\frac{\partial v_k}{\partial v_1}\right)$ **Near** $\partial\Omega$

Assume Ω is convex as in (4). We now compute $\det(\frac{\partial v_k}{\partial v_1})$ for a carefully chosen specular back-time cycle near the boundary $\partial\Omega$. We assume $x_1 \in \partial\Omega$. Given ε_0 small, we choose v_1 such that

$$|v_1| = \varepsilon_0, \quad v_1 \cdot n(x_1) = \frac{v \cdot \nabla\xi(x_1)}{|\nabla\xi(x_1)|} = \varepsilon_0^2. \tag{161}$$

We shall analyze the specular back-time cycle of $(0, x_1, v_1)$: (t_k, x_k, v_k). Letting $s_k = t_{\mathbf{b}}(x_k, v_k)$, we have $\xi(x_1 - s_1 v_1) = 0$, $x_2 = x_1 - s_1 v_1 \in \partial\Omega$ and for $k \geq 2$:

$$\xi(x_1 - \sum_{j=1}^{k} s_j v_j) = 0, \quad v_k = R(x_k)v_{k-1}, \quad x_k = x_{k-1} - s_k v_k \in \partial\Omega.$$

Proposition 33 *For any finite $k \geq 1$,*

$$\frac{\partial v_k^i}{\partial v_1^l} = \delta_{li} + \zeta(k)n^i(x_1)n^l(x_1) + O(\varepsilon_0), \tag{162}$$

where O depends on k, and ζ is defined as $\zeta(1) = 0$,

$$\zeta(k) = 4\sum_{p=1}^{k-2}(-1)^{k-p+1} + 4\sum_{p=1}^{k-2}(-1)^{k-1-p}\zeta(p) + 2 + 3\zeta(k-1), \ \text{for } k \geq 2. \tag{163}$$

In particular, $\zeta(k)$ is an even integer so that

$$\det\left(\frac{\partial v_k^i}{\partial v_1^l}\right) = \{\zeta(k) + 1\} + O(\varepsilon_0) \neq 0.$$

Proof. From $n(x_j) = \frac{\nabla\xi(x_j)}{|\nabla\xi(x_j)|}$, since $v_j = v_{j-1} - 2\{n(x_j) \cdot v_{j-1}\}n(x_j)$, we define

$$d_j \equiv v_j \cdot \nabla\xi(x_j) = -v_{j-1} \cdot \nabla\xi(x_j) \geq 0. \tag{164}$$

By the Velocity Lemma 5 and our choice of v_1 in (161), if $\sum_{j=1}^{k-1} s_j < C$, we have $C_1\alpha(0) \leq \alpha(\sum_{p=1}^{j} s_p) \leq C_2\alpha(0)$, for all $j = 1, 2, ..., k-1$. But $\alpha(0) = \{v_1 \cdot \nabla\xi(x_1)\}^2 \backsim \varepsilon_0^4$ and $\alpha(\sum_{p=1}^{j} s_p) = \{v_j \cdot \nabla\xi(x_j)\}^2$, we then have

$$v_j \cdot n(x_j) = -v_{j-1} \cdot n(x_j) \backsim C\varepsilon_0^2. \tag{165}$$

We therefore deduce that, by denoting $n_j = n(x_j)$,

$$\begin{aligned}
|v_j - v_1| &\leq |v_j - v_{j-1}| + |v_{j-1} - v_{j-2}| + ... + |v_2 - v_1| \\
&\leq 2|v_{j-1} \cdot n_j| + 2|v_{j-2} \cdot n_{j-1}| + ... + 2|v_1 \cdot n_2| \\
&\leq 2jC\varepsilon_0^2.
\end{aligned} \tag{166}$$

With the assumption $\sum_j^{k-1} s_j < C$, by $|v_1| = \varepsilon_0$, we deduce that

$$|x_k - x_1| \leq C\sum_{j=1}^{k-1}|v_j| \leq C_k\varepsilon_0. \tag{167}$$

We first estimate the next s_k. Note that for $k \geq 2$,

$$\xi(x_k + s_{k-1}v_{k-1}) = 0, \qquad \xi(x_k - s_k v_k) = 0.$$

We then use Taylor expansion at x_k to get

$$\begin{aligned}
\xi(x_k + s_{k-1}v_{k-1}) &= \xi(x_k) + s_{k-1}v_{k-1} \cdot \nabla\xi(x_k) + \frac{1}{2}s_{k-1}^2 v_{k-1}\nabla^2\xi(x_k)v_{k-1} + O(s_{k-1}^3 v_{k-1}^3); \\
\xi(x_k - s_k v_k) &= \xi(x_k) - s_k v_k \cdot \nabla\xi(x_k) + \frac{1}{2}s_k^2 v_k \nabla^2\xi(x_k)v_k + O(s_k^3 v_k^3).
\end{aligned}$$

But $\frac{\nabla \xi(x_k)}{|\nabla \xi(x_k)|} = n_k$, $\xi(x_k) = 0$, $|v_k| = |v_{k+1}| = O(\varepsilon_0)$, by (164), (165) and (166) , we have

$$1 - \frac{1}{2}s_{k-1}\frac{v_{k-1}\nabla^2\xi(x_k)v_{k-1}}{d_k} + O(\varepsilon_0)s_{k-1}^2 = 0,$$

$$1 - \frac{1}{2}s_k\frac{v_k\nabla^2\xi(x_k)v_k}{d_k} + O(\varepsilon_0)s_k^2 = 0,$$

where $d_k \backsim \varepsilon_0^2$. Therefore, by (166) and (167),

$$s_{k-1} = \frac{2d_k}{v_{k-1}\nabla^2\xi(x_k)v_{k-1}} + O(\varepsilon_0) = \frac{2d_k}{v_1\nabla^2\xi(x_1)v_1} + O(\varepsilon_0),$$

$$s_k = \frac{2d_k}{v_k\nabla^2\xi(x_k)v_k} + O(\varepsilon_0) = \frac{2d_k}{v_1\nabla^2\xi(x_1)v_1} + O(\varepsilon_0), \tag{168}$$

so that $s_{k+1} - s_k = O(\varepsilon_0)$, for finite k.

We now compute from $v_k = v_{k-1} - 2\{n(x_k) \cdot v_{k-1}\}n(x_k)$,

$$\frac{\partial v_k^i}{\partial v_1^l} = \frac{\partial v_{k-1}^i}{\partial v_1^l} - 2(v_{k-1} \cdot n_k)\partial_{v_1^l}n_k^i - 2(v_{k-1} \cdot \partial_{v_1^l}n_k)n_k^i - 2(\frac{\partial v_{k-1}}{\partial v_1^l} \cdot n_k)n_k^i. \tag{169}$$

To compute $\partial_{v_1^l}n_k^m$ in (169), we note $n^m(y) = \frac{\partial_m \xi(y)}{|\nabla \xi(y)|}$ and

$$\partial_{v_1^l}n_k^m = \partial_{v_1^l}\{n^m(x_1 - \sum_{j=1}^{k-1}s_jv_j)\}$$

$$= \partial_q n^m(x_k) \times \{-\sum_{j=1}^{k-1}\partial_{v_1^l}s_jv_j^q - \sum_{j=1}^{k-1}s_j\frac{\partial v_j^q}{\partial v_1^l}\}$$

$$= (\frac{\partial_{mq}\xi}{|\nabla\xi|} - \frac{n^m\partial_{oq}\xi n^o}{|\nabla\xi|})|_{x_k} \times \{-\sum_{j=1}^{k-1}\partial_{v_1^l}s_jv_j^q - \sum_{j=1}^{k-1}s_j\frac{\partial v_j^q}{\partial v_1^l}\}. \tag{170}$$

To compute $\partial_{v_1^l}s_j$, we recall $\xi(x_{j+1}) = \xi(x_j) = 0$ so that

$$\xi(x_1 - \sum_{p=1}^{j}s_pv_p) = 0, \quad \xi(x_1 - \sum_{p=1}^{j-1}s_pv_p) = 0.$$

Taking their v_1^l derivatives, we split $\sum_{p=1}^{j}$ into $\sum_{p=1}^{j-1} + \sum_{p=j}^{j}$ to get

$$\sum_o \partial_o\xi(x_{j+1})\{-\sum_{p=1}^{j-1}\frac{\partial v_p^o}{\partial v_1^l}s_p - \sum_{p=1}^{j-1}v_p^o\partial_{v_1^l}s_p\} = \sum_o \partial_o\xi(x_{j+1})\{\frac{\partial v_j^o}{\partial v_1^l}s_j + v_j^o\partial_{v_1^l}s_j\},$$

$$\sum_o \partial_o\xi(x_j)\{-\sum_{p=1}^{j-1}\frac{\partial v_p^o}{\partial v_1^l}s_p - \sum_{p=1}^{j-1}v_p^o\partial_{v_1^l}s_p\} = 0.$$

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Subtracting these two identities, we deduce

$$\sum_o \partial_o \xi(x_{j+1}) v_j^o \partial_{v_1^l} s_j = -\sum_o \partial_o \xi(x_{j+1}) \frac{\partial v_j^o}{\partial v_1^l} s_j$$

$$+ \sum_o \{\partial_o \xi(x_{j+1}) - \partial_o \xi(x_j)\} \{-\sum_{p=1}^{j-1} \frac{\partial v_p^o}{\partial v_1^l} s_p - \sum_{p=1}^{j-1} v_p^o \partial_{v_1^l} s_p\}.$$

But by the Taylor expansion and (167),

$$\partial_o \xi(x_{j+1}) - \partial_o \xi(x_j) = \partial_{oe} \xi(x_j)(x_{j+1}^e - x_j^e) + O(|x_{j+1} - x_j|^2)$$
$$= -\partial_{oe} \xi(x_j) s_j v_j^e + O(\varepsilon_0^2).$$

Rewriting $\sum_o \partial_o \xi(x_{j+1}) v_j^o = v_j \cdot \nabla \xi(x_{j+1})$, we therefore have

$$v_j \cdot \nabla \xi(x_{j+1}) \partial_{v_1^l} s_j = -\sum_o \partial_o \xi(x_{j+1}) \frac{\partial v_j^o}{\partial v_1^l} s_j$$

$$+ \sum_{o,e} \partial_{oe} \xi(x_j) s_j v_j^e \{\sum_{p=1}^{j-1} \frac{\partial v_p^o}{\partial v_1^l} s_p + \sum_{p=1}^{j-1} v_p^o \partial_{v_1^l} s_p\}$$

$$+ O(\varepsilon_0^2) \{\sum_{p=1}^{j-1} \frac{\partial v_p^0}{\partial v_1^l} s_p + \sum_{p=1}^{j-1} v_p^0 \partial_{v_1^l} s_p\}.$$

Since $v_j \cdot \nabla \xi(x_{j+1}) = -d_{j+1}$, from (168), $\sum_{o,e} \frac{\partial_{oe} \xi(x_j) s_j v_j^e v_p^o}{-d_{j+1}} = -2 + O(\varepsilon_0)$ and

$$\partial_{v_1^l} s_j = \sum_o \frac{\partial_o \xi(x_{j+1})}{d_{j+1}} \frac{\partial v_j^o}{\partial v_1^l} s_j - \{2 - O(\varepsilon_0)\} \sum_{p=1}^{j-1} \partial_{v_1^l} s_p - \sum_{o,e} \sum_{p=1}^{j-1} \frac{\partial_{oe} \xi(x_j) s_j v_j^e}{d_{j+1}} \frac{\partial v_p^o}{\partial v_1^l} s_p$$

$$+ O(1) \{\sum_{p=1}^{j-1} \frac{\partial v_p^o}{\partial v_1^l} s_p + \sum_{p=1}^{j-1} v_p^o \partial_{v_1^l} s_p\}. \qquad (171)$$

We first claim that for $1 \leq j \leq k$

$$|\partial_{v_1^l} s_j| \leq \frac{C_k}{\varepsilon_0^2}, \quad |\frac{\partial v_j^0}{\partial v_1^l}| \leq C_k. \qquad (172)$$

We shall prove this via an induction of j. In fact, when $j = 1$, $\frac{\partial v_1^0}{\partial v_1^l} = \delta_{ol}$, and from $\xi(x_1 - s_1 v_1) = 0$, we deduce

$$\partial_{v_1^l} s_1 = \frac{\partial_l \xi(x_2) s_1}{d_2} = O(\varepsilon_0^{-2}). \qquad (173)$$

And a simple induction leads to the desired result (172).

From (172) and (171), we have

$$\partial_{v_1^l} s_j = \sum_o \frac{\partial_o \xi(x_{j+1})}{d_{j+1}} \frac{\partial v_j^o}{\partial v_1^l} s_j - 2\sum_{p=1}^{j-1} \partial_{v_1^l} s_p + O(\frac{1}{\varepsilon_0}).$$

By letting $A_j \equiv \sum_{p=1}^j \partial_{v_1^l} s_p$ and moving one copy of $\sum_{p=1}^{j-1} \partial_{v_1^l} s_p$ to the right hand side, we deduce

$$A_j = \sum_o \frac{\partial_o \xi(x_{j+1})}{d_{j+1}} \frac{\partial v_j^o}{\partial v_1^l} s_j - A_{j-1} + O(\frac{1}{\varepsilon_0})$$

so that we can obtain explicit formula for A_j as

$$
\begin{aligned}
A_j &= \sum_o \frac{\partial_o \xi(x_{j+1})}{d_{j+1}} \frac{\partial v_j^o}{\partial v_1^l} s_j - A_{j-1} + O(\frac{1}{\varepsilon_0}) \\
&= \sum_o \frac{\partial_o \xi(x_{j+1})}{d_{j+1}} \frac{\partial v_j^o}{\partial v_1^l} s_j - \sum_o \frac{\partial_o \xi(x_j)}{d_j} \frac{\partial v_{j-1}^o}{\partial v_1^l} s_{j-1} + A_{j-2} + O(\frac{1}{\varepsilon_0})... \\
&= \sum_{p=1}^j \sum_o (-1)^{j-p} \frac{\partial_o \xi(x_{p+1})}{d_{p+1}} \frac{\partial v_p^o}{\partial v_1^l} s_p + O(\frac{1}{\varepsilon_0}), \qquad (174)
\end{aligned}
$$

we have used the fact by (173), $A_1 = \partial_{v_1^l} s_1 = \sum_o \frac{\partial_o \xi(x_2)}{d_2} \frac{\partial v_1^o}{\partial v_1^l} s_1$. Now finally we recall (170), $\partial_{v_1^l} n_k^i = O(\frac{1}{\varepsilon_0})$, so that the second term on the right hand side in (169) is of the order $O(\varepsilon_0)$. Hence

$$
\begin{aligned}
\frac{\partial v_k^i}{\partial v_1^l} &= \frac{\partial v_{k-1}^i}{\partial v_1^l} - 2(v_{k-1}^m (\frac{\partial_{mq}\xi}{|\nabla \xi|} - \frac{n^m \partial_{oq}\xi n^o}{|\nabla \xi|})|_{x_k} \times \{-\sum_{j=1}^{k-1} \partial_{v_1^l} s_j v_j^q - \sum_{j=1}^{k-1} s_j \frac{\partial v_j^q}{\partial v_1^l}\}) n_k^i \\
&\quad -2(\frac{\partial v_{k-1}^i}{\partial v_1^l} \cdot n_k) n_k^i + O(\varepsilon_0).
\end{aligned}
$$

Since $\sum_m v_{k-1}^m n^m(x_k) = O(\varepsilon_0^2)$, the second term on the right hand side is

$$
\begin{aligned}
&2\sum_{j=1}^{k-1} \frac{v_{k-1}^m \partial_{mq}\xi(x_k) v_j^q}{|\nabla \xi(x_k)|} \times \partial_{v_1^l} s_j n_k^i - 2(\frac{\partial v_{k-1}}{\partial v_1^l} \cdot n_k) n_k^i + O(\varepsilon_0) \\
&= 2\frac{v_1^m \partial_{mq}\xi(x_1) v_1^q}{|\nabla \xi(x_1)|} \times \{\sum_{j=1}^{k-1} \partial_{v_1^l} s_j\} n_1^i - 2(\frac{\partial v_{k-1}}{\partial v_1^l} \cdot n_1) n_1^i + O(\varepsilon_0) \text{ by (167)} \\
&= 2\frac{v_1^m \partial_{mq}\xi(x_1) v_1^q}{|\nabla \xi(x_1)|} \times \sum_{j=1}^{k-1} (-1)^{k-p-1} \frac{\partial_o \xi(x_{p+1})}{d_{p+1}} \frac{\partial v_p^o}{\partial v_1^l} s_p n_1^i + O(\varepsilon_0) \text{ by (174)}.
\end{aligned}
$$

Note $\frac{s_p}{d_{p+1}} v_1^m \partial_{mq}\xi(x_1) v_1^q = 4 + O(\varepsilon_0)$ from (164) and (168), we deduce

$$\frac{\partial v_k^i}{\partial v_1^l} = \frac{\partial v_{k-1}^i}{\partial v_1^l} + 4\sum_{j=1}^{k-1} (-1)^{k-p-1} n_1^o \frac{\partial v_p^o}{\partial v_1^l} n_1^i - 2(\frac{\partial v_{k-1}^i}{\partial v_1^l} \cdot n_1) n_1^i + O(\varepsilon_0).$$

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Clearly, $\xi(1) = 0$, and assume (162) is valid up to $k - 1$. Then

$$
\begin{aligned}
\frac{\partial v_k^i}{\partial v_1^l} &= \delta_{li} + \zeta(k-1)n_1^i n_1^l + 4\sum_{p}^{k-1}(-1)^{k-p-1}n_1^o(\delta_{ol} + \zeta(p)n_1^o n_1^l)n_1^i \\
&\quad -2(\delta_{lo} + \zeta(k-1)n_1^o n_1^l) \cdot n_1^o n_1^i + O(\varepsilon_0).
\end{aligned}
$$

Notice that $\sum_o n_1^o n_1^o = 1$, and splitting $\sum_j^{k-1} = \sum_j^{k-2} + \sum_{j=k-1}^{k-1}$ and collecting terms, we conclude our proposition. ∎

4.3.3 L^∞ Decay for $U(t)$

We now fix any point (t, x, v) so that $(x, v) \notin \gamma_0$. Let the back-time specular cycle of (t, x, v) be $[X_{\mathbf{cl}}(s_1), V_{\mathbf{cl}}(s_1)]$. By (31), we use twice (159) to derive $h(t, x, v) =$

$$
\begin{aligned}
&e^{-\nu(v)t}h_0(X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0)) \\
&+ \int_0^t e^{-\nu(v)(t-s_1)} \int K_w(V_{\mathbf{cl}}(s_1), v')e^{-\nu(v')s_1}h_0\left(X_{\mathbf{cl}}'(0), V_{\mathbf{cl}}'(0)\right)dv' \qquad (175)\\
&+ \int_0^t \int_0^{s_1} \int e^{-\nu(v)(t-s_1)-\nu(v')(s_1-s)}K_w(V_{\mathbf{cl}}(s_1), v')K_w(V_{\mathbf{cl}}'(s), v'')h\left(X_{\mathbf{cl}}'(s), v''\right).
\end{aligned}
$$

where the back-time specular cycle from $(s_1, X_{\mathbf{cl}}(s_1), v')$ is denoted by

$$
X_{\mathbf{cl}}'(s) = X_{\mathbf{cl}}(s; s_1, X_{\mathbf{cl}}(s_1), v'), \qquad V_{\mathbf{cl}}'(s) = V_{\mathbf{cl}}(s; s_1, X_{\mathbf{cl}}(s_1), v'). \qquad (176)
$$

More explicitly, let t_k and $t_{k'}'$ be the corresponding times for both specular cycles as in (157). For $t_{k+1} \leq s_1 < t_k$, $t_{k'+1}' \leq s < t_{k'}'$

$$
X_{\mathbf{cl}}'(s) = X_{\mathbf{cl}}(s; s_1, X_{\mathbf{cl}}(s_1), v') \equiv x_{k'}' + (s - t_{k'}')v_{k'}' \qquad (177)
$$

where $x_{k'}' = X_{\mathbf{cl}}(t_{k'}'; s_1, x_k + (s_1 - t_k)v_k, v')$, $v_{k'}' = V_{\mathbf{cl}}(t_{k'}'; s_1, x_k + (s_1 - t_k)v_k, v')$. Recall α in (36) and define naturally

$$
\alpha(x, v) \equiv \alpha(t) = \xi^2(x) + [v \cdot \nabla\xi(x)]^2 - [v\nabla^2\xi(x)v]\xi(x). \qquad (178)
$$

We define the main set

$$
A_\alpha = \{(x, v) : x \in \bar{\Omega}, \ \frac{1}{N} \leq |v| \leq N, \ \text{and} \ \alpha(x, v) \geq \frac{1}{N}\}. \qquad (179)
$$

We remark that for x is near $\partial\Omega$, $\det\{\frac{\partial v_2}{\partial v_1}\} \backsim 3$ in Lemma 33 for v_1 is almost tangential to $n(x)$. On the other hand, by (39), it is easy to compute for $v_1 = n(x)$, $\det\{\frac{\partial v_2}{\partial v_1}\} \backsim -1$ since $t_{\mathbf{b}} \backsim 0$. This implies from continuity that there is v_1 such that $\det\{\frac{\partial v_2}{\partial v_1}\} = 0$ even after one specular reflection. However, as shown next, such a zero set is small if Ω is both analytic and convex.

Lemma 34 *Fix k and k'. Define for $t_{k+1} \leq s_1 \leq t_k, s \in \mathbf{R}$*

$$
J \equiv J_{k,k'}(t, x, v, s_1, s, v') \equiv \det\left(\frac{\partial\{x_{k'}' + (s - t_{k'}')v_{k'}'\}}{\partial v'}\right).
$$

For any $\varepsilon > 0$ sufficiently small, there is $\delta(N, \varepsilon, T_0, k, k') > 0$ and an open covering $\cup_{i=1}^m B(t_i, x_i, v_i; r_i)$ of $[0, T_0] \times A_\alpha$ and corresponding open sets O_{t_i, x_i, v_i} for $[t_{k+1} + \varepsilon, t_k - \varepsilon] \times \mathbf{R} \times \mathbf{R}^3$ with $|O_{t_i, x_i, v_i}| < \varepsilon$, such that

$$|J_{k,k'}(t, x, v, s_1, s, v')| \geq \delta > 0,$$

for $0 \leq t \leq T_0$, $(x, v) \in A_\alpha$ and (s_1, s, v') in

$$O^c_{t_i, x_i, v_i} \cap [t_{k+1} + \varepsilon, t_k - \varepsilon] \times [0, T_0] \times \{|v'| \leq 2N\}.$$

Proof. Fix (t, x, v) such that $x, v \in A_\alpha$ in (179), and fix k, k'. Since $x, v \notin \gamma_0$, by Velocity Lemma 5, we deduce that $\alpha(t_k) = \{n(x_k) \cdot v_k\}^2 \neq 0$ so that $t_k - t_{k+1} > 0$ from (40). We note since $|v'| \geq \frac{1}{N}$ from (179), $t_k - t_{k+1} \leq N \text{diam}\Omega$. Since for $t_{k+1} + \frac{\varepsilon}{2} \leq s_1 \leq t_k - \frac{\varepsilon}{2}$, $x_k - (s_1 - t_k)v_k \in \Omega$, the interior of the domain with ε sufficiently small. From (178) $\alpha(x_k + (s_1 - t_k)v_k, v') > 0$ for all v'. This implies that along its back-time specular cycle $[X'_{\mathbf{cl}}(s), V'_{\mathbf{cl}}(s)]$, $\alpha(t'_{l'}) > 0$ and $v'_{l'} \cdot n(x'_{l'}) \neq 0$ from the Velocity Lemma 5. Clearly, by the Velocity Lemma 5 and part (2) of Lemma 6, t'_l, x'_l, v'_l are analytical functions of s_1, s, v'. Therefore the function $J_{k,k'}(t, x, v, s_1, s, v')$ is well-defined, and analytic for all $v' \in \mathbf{R}^3$, $s \in \mathbf{R}$, and $t_{k+1} + \frac{\varepsilon}{2} \leq s_1 \leq t_k - \frac{\varepsilon}{2}$. Moreover, expanding as a polynomial of s, we obtain

$$J(t, x, v, s_1, s, v') = \det\left(\frac{\partial v'_{k'}}{\partial v'}\right) s^3 + p_1 s^2 + p_2 s + p_3$$

where $p_i = p_i(t, x, v, s_1, v')$ is an analytical function of $(s_1, v') \in (t_{k+1} + \frac{\varepsilon}{2}, t_k - \frac{\varepsilon}{2}) \times \mathbf{R}^3$. But at $s_1 = t_{k+1}$, $X_{\mathbf{cl}}(s_1) = x_{k+1} \in \partial\Omega$. From Proposition 33, there exists v'_0 with $\alpha(t_{k+1}) = \{v'_0 \cdot n(x_{k+1})\}^2 = \varepsilon_0^4 > 0$ such that $\det\left(\frac{\partial v'_{k'}}{\partial v'}\right)|_{v'=v_0} \neq 0$. Since $v_0 \cdot n(x_{k+1}) \neq 0$, by the Velocity lemma 5 and Lemma 6 for such v'_0, $\det\left(\frac{\partial v'_k}{\partial v'}\right)$ is continuous with respect to y near x_{k+1}. In particular, along $[X'_{\mathbf{cl}}(s), V'_{\mathbf{cl}}(s)]$ in (176), $\det\left(\frac{\partial v'_{k'}}{\partial v'}\right)|_{v'=v_0} \neq 0$ for some s_1 at $t_{k+1} + \frac{3\varepsilon}{4}$, for ε sufficiently small so that $x_k + (s_1 - t_k)v_k \backsim x_k + (t_{k+1} - t_k)v_k \backsim x_{k+1}$. Therefore, $\det\left(\frac{\partial v'_k}{\partial v'}\right)$ is an analytical function which is not identically zero, so is $J_{k,k'}(t, x, v, s_1, s, v')$ as an analytical function of $(s_1, s, v') \in (t_{k+1} + \frac{\varepsilon}{2}, t_k - \frac{\varepsilon}{2}) \times \mathbf{R} \times \mathbf{R}^3$. By Lemma 8, for each (t, x, v), there exists an open set $O_{t,x,v}$ of s_1, s, v' in $(t_{k+1} + \frac{\varepsilon}{2}, t_k - \frac{\varepsilon}{2}) \times \mathbf{R} \times \mathbf{R}^3$ such that $|O_{t,x,v}| < \varepsilon$, and for $(s_1, s, v') \notin O_{t,x,v}$, $J(t, x, v, s_1, s, v') \neq 0$. Therefore, by continuity of $J(t, x, v, s_1, s, v')$ with respect to s_1, s, v', there exists $\delta_{t,x,v,N,T_0,\varepsilon,\varepsilon_1,k,k'} > 0$, such that

$$|J(t, x, v, s_1, s, v')| > \delta_{t,x,v,N,T_0,\varepsilon,k,k'} > 0$$

for the compact set:

$$(s_1, s, v') \in O^c_{t,x,v} \cap [t_{k+1} + \frac{3\varepsilon}{4}, t_k - \frac{3\varepsilon}{4}] \times [0, T_0] \times \{|v'| \leq 2N\}.$$

Since $\alpha(x, v) \geq \frac{1}{N}$, by the Velocity Lemma 5 and par (2) of Lemma 6, t_k, x_k, and v_k are analytic functions respect to (t, x, v), and $x'_{k'}$ and $t'_{k'}$ are analytic with

respect to (t, x, v) as well. Therefore, there exists an open ball $B(t, x, v; r_{(t,x,v,\varepsilon)})$ such that if $(\tau, y, w) \in B(t, x, v; r_{(t,x,v,\varepsilon)})$,

$$
\begin{aligned}
t_{k+1}(\tau, y, w) &< t_{k+1}(t, x, v) + \frac{\varepsilon}{2} < s_1 < t_k(t, x, v) - \frac{\varepsilon}{2} < t_k(\tau, y, w), \\
t_{k+1}(t, x, v) + \frac{\varepsilon}{2} &< t_{k+1}(\tau, y, w) + \varepsilon, \quad t_k(\tau, y, w) - \varepsilon < t_k(t, x, v) - \frac{\varepsilon}{2} (180)
\end{aligned}
$$

Hence by (180), $J_{k,k'}(\tau, y, z, s_1, s, v')$ is well-defined, continuous, and we may then assume

$$
|J_{k,k'}(\tau, y, z, s_1, s, v')| > \frac{\delta_{t,x,v,T_0,N,\varepsilon,k,k'}}{2} > 0,
$$

in $B(t, x, v; r_{(t,x,v,\varepsilon)}) \times O_{t,x,v}^c \cap [t_{k+1}(t, x, v) + \frac{\varepsilon}{2}, t_k(t, x, v) - \frac{\varepsilon}{2}] \times [0, T_0] \times \{|v'| \leq 2N\}$, and clearly also on the smaller set (by (180)):

$$
B(t, x, v; r_{(t,x,v,\varepsilon)}) \times O_{t,x,v}^c \cap [t_{k+1}(\tau, y, z) + \varepsilon, t_k(\tau, y, z) - \varepsilon] \times [0, T_0] \times \{|v'| \leq 2N\}.
$$

Now by a finite covering for the compact set $[0, T_0] \times A_\alpha$ by such $B(t, x, v; r)$, there are $[t_1, x_1, v_1], ... [t_m, x_m, v_m]$ such that $[0, T_0] \times A_\alpha \subset \cup_{i=1}^m B(t_i, x_i, v_i; r_i)$. For any point (t, x, v), there is i so $(t, x, v) \in B(t_i, x_i, v_i; r_i(\varepsilon))$ and

$$
|J(t, , x, v, s_1, s, v')| > \min_{1 \leq i \leq m} \frac{\delta_{i,T_0,N,\varepsilon.k,k'}}{2} > 0
$$

for $(s_1, s, v') \in O_{t_i, x_i, v_i}^c \cap [t_{k+1} + \varepsilon, t_k - \varepsilon] \times [0, T_0] \times \{|v'| \leq 2N\}$. ∎

Theorem 35 *Assume $w^{-2}\{1 + |v|\} \in L^1$. Assume that ξ is both strictly convex (4) and analytic, and the mass (17) and energy (18) are conserved. In the case of Ω has rotational symmetry (5), we also assume conservation of corresponding angular momentum (19). Let $h_0 \in L^\infty$. There exits a unique solution to both the (23) and (27) with boundary specular condition, and the exponential decay (142) is valid.*

Proof. The well-posedness follows from the exact argument in the proof of Theorem 28. Thanks to Lemma 29, we only need to establish the finite time estimate (143). Recall A_α in (179).

STEP 1: Estimate of $h(t, x, v)\mathbf{1}_{A_\alpha}$. We first express and estimate the main part $h(t, x, v)\mathbf{1}_A$ through (175). As in the case of bounce-back reflection, the first and the second terms in (175) are bounded by (144) and (145a) respectively.

For the third main contribution in (175), notice that along the back-time specular cycles $[X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s)]$ and $[X'_{\mathbf{cl}}(s), V'_{\mathbf{cl}}(s)]$ in (176), $|V_{\mathbf{cl}}(s_1)| \equiv |v|$ and $|V'_{\mathbf{cl}}(s_1)| \equiv |v'|$. Hence, the integration over $|v'| \geq 2N$ or $|v'| \leq 2N$ but $|v''| \geq 3N$ are bounded by (147). By using the same approximation, we only need to concentrate on the bounded set $\{|v'| \leq 2N$ and $|v''| \leq 3N\}$ as in (148) of

$$
\int_0^t \int_0^{s_1} \int_{|v'| \leq 2N, |v''| \leq 3N} e^{-v_0(t-s)} \mathbf{1}_{A_\alpha} |h(s, X'_{\mathbf{cl}}(s), v'')| dv' dv'' ds_1 ds
$$

$$
= \int_{\substack{\alpha(X_{\mathbf{cl}}(s_1), v') < \varepsilon \\ |v'| \leq 2N, |v''| \leq 3N,}} + \int_{\substack{\alpha(X_{\mathbf{cl}}(s_1), v') \geq \varepsilon \\ |v'| \leq 2N, |v''| \leq 3N,}}
$$

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where we have further spit into $\alpha(X_{\mathbf{cl}}(s_1), v') \le \varepsilon$ and $\alpha(X_{\mathbf{cl}}(s_1), v') > \varepsilon$.

In the case $\alpha(X_{\mathbf{cl}}(s_1), v') \le \varepsilon$, $\xi^2(X_{\mathbf{cl}}(s_1)) + [v' \cdot \nabla\xi(X_{\mathbf{cl}}(s_1))]^2 \le \varepsilon$. Hence for ε small, $X_{\mathbf{cl}}(s_1) \backsim \partial\Omega$, and $|\nabla\xi(X_{\mathbf{cl}}(s_1))| \ge \frac{1}{2}$. The first part integral is bounded by

$$C_N \int_0^t \int_0^{s_1} e^{-v_0(t-s)} ||h(s)||_\infty ds ds_1 \int_{\substack{\alpha(X_{\mathbf{cl}}(s_1), v') \le \varepsilon \\ |v'| \le 2N, |v''| \le 3N,}}$$

$$\le \quad C_N \sup_{t \ge s} e^{-\frac{v_0}{2}(t-s)} ||h(s)||_\infty \int_{|v' \cdot \frac{\nabla\xi(X_{\mathbf{cl}}(s_1))}{|\nabla\xi(X_{\mathbf{cl}}(s_1))|}| \le 2\varepsilon, |v'| \le 2N, |v''| \le 3N} \quad \le C_N \varepsilon \sup_{t \ge s} e^{-\frac{v_0}{2}(t-s)} ||h(s)||_\infty.$$

Finally, from (177), we bound the first main term $\alpha(X_{\mathbf{cl}}(s_1), v') \ge \varepsilon$ as

$$C_N \int_0^t e^{-\nu_0(t-s)} \int_0^{s_1} \int_{\substack{\alpha(X_{\mathbf{cl}}(s_1), v') \ge \varepsilon \\ |v'| \le 2N, |v''| \le 3N}} \mathbf{1}_{A_\alpha} |h(s, X'_{\mathbf{cl}}(s), v'')| dv' dv''$$

$$= \quad C_N \sum_{k, k'} \int_{t_{k+1}}^{t_k} \int_{t'_{k'+1}}^{t'_{k'}} \int_{\substack{\alpha(X_{\mathbf{cl}}(s_1), v') \ge \varepsilon \\ |v'| \le 2N, |v''| \le 3N}} \mathbf{1}_{A_\alpha} e^{-\nu_0(t-s)} |h(s, x'_{k'} + (s - t'_{k'})v'_{k'}, v'')|,$$

where $[t'_{k'}, x'_{k'}, v'_{k'}]$ is the back-time cycle of $(s_1, x_k + (s_1 - t_k)v_k, v_k)$, for $t_{k+1} \le s_1 \le t_k$.

We now study $x'_{k'} + (s - t'_{k'})v'_{k'}$. By the repeatedly using Velocity Lemma 5, we deduce for $(t, x, v) \in A_\alpha$ and $\alpha(X(s_1), v') \ge \varepsilon, |v'| \le 2N$:

$$\alpha(t_l) = \{v_l \cdot n_{x_l}\}^2 \ge e^{-\{C_\xi N - 1\}T_0} \alpha(s_1) \ge C_{T_0, \xi, N} > 0;$$

$$\alpha(t'_l) = \{v'_l \cdot n_{x'_l}\}^2 \ge e^{-\{C_\xi N - 1\}T_0} \alpha(X_{\mathbf{cl}}(s_1), v') \ge C_{T_0, N, \xi} \varepsilon > 0.$$

Therefore, applying (40) in Lemma 6 yields $t_l - t_{l+1} \ge \frac{c_{T_0, \xi, N}}{N^2}$ and $t'_l - t'_{l+1} \ge \frac{c_{T_0, \xi, N}\varepsilon}{4N^2}$ so that

$$k \le \frac{T_0 N^2}{c_{T_0, \xi, N}} = C_{T_0, \xi, N}, \quad k' \le \frac{T_0 N^2}{c_{T_0, \xi, N}\varepsilon} = C_{T_0, \xi, N, \varepsilon}. \quad (181)$$

We therefore further split the s_1-integral as

$$C_{K, N} \int_{t_{k+1}}^{t_k} \int_{|v'| \le 2N, |v''| \le 3N} \sum_{k \le C_{T_0, N}, k' \le C_{T_0, N, \varepsilon}} \int_{t'_{k'+1}}^{t'_{k'}} \mathbf{1}_{A_\alpha} e^{-\nu_0(t-s)} |h(s, x'_{k'} + (s - t'_{k'})v'_{k'}, v'')|$$

$$= \int_{t_{k+1}+\varepsilon}^{t_k - \varepsilon} + \int_{t_k}^{t_k - \varepsilon} + \int_{t_{k+1}}^{t_{k+1}+\varepsilon}.$$

Since $\sum_{k'} \int_{t'_{k'+1}}^{t'_{k'}} = \int_0^{s_1}$, the last two terms make small contribution as

$$\varepsilon C_{K, N} \sup_{0 \le s \le t} e^{-\nu_0(t-s)} ||h(s)||_\infty \int_0^{T_0} \int_{|v'| \le 2N, |v''| \le 3N} = \varepsilon C_{K, N} \sup_{0 \le s \le t} e^{-\nu_0(t-s)} ||h(s)||_\infty.$$

For the main contribution $\int_{t_{k+1}+\varepsilon}^{t_k-\varepsilon}$, By Lemma 34, on the set $O_{t_i,x_i,v_i}^c \cap [t_{k+1} + \varepsilon, t_k - \varepsilon] \times [0, T_0] \times \{|v'| \le N\}$, we can define a change of variable

$$y \equiv x'_{k'} + (s - t'_{k'})v'_{k'},$$

so that $\det(\frac{\partial y}{\partial v'}) > \delta$ on the same set. By the Implicit Function Theorem, there are an finite open covering $\cup_{j=1}^m V_j$ of $O_{t_i,x_i,v_i}^c \cap [t_{k+1}+\varepsilon, t_k-\varepsilon] \times [0, T_0] \times \{|v'| \le N\}$, and smooth function F_j such that $v' = F_j(t, x, v, y, s_1, s)$ in V_j. We therefore have

$$\sum_{k,k'} \int_{t_{k+1}+\varepsilon}^{t_k-\varepsilon} \int_{|v'|\le 2N,} \int_{t'_{k'+1}}^{t'_{k'}} \le \sum_{k,k'} \int_{t_{k+1}+\varepsilon}^{t_k-\varepsilon} \int_{|v'|\le 2N,} \int_{t'_{k'+1}}^{t'_{k'}} \mathbf{1}_{O_{t_i,x_i,v_i}} + \sum_{j,k,k'} \int_{t_{k+1}+\varepsilon}^{t_k-\varepsilon} \int_{|v'|\le 2N,} \int_{t'_{k'+1}}^{t'_{k'}} \mathbf{1}_{V_j}.$$

Since $\sum_{k'} \int_{t'_{k'+1}}^{t'_{k'}} = \int_0^{s_1} \le \int_0^{T_0}$ and $|O_{t_i,x_i,v_i}| < \varepsilon$, the first part is bounded by $C_N \varepsilon e^{-\frac{\nu_0}{2}t} \sup_s\{e^{\frac{\nu_0}{2}s}||h(s)||_\infty\}$ from Lemma 34.

For the second part, we can make a change of variable $v' \to y = x'_{k'} + (s - t'_{k'})v'_{k'}$ on each V_j to get

$$C_{\varepsilon,T_0,N} \sum_{j,k,k'} \int_{V_j} \int_{|v''|\le 3N} e^{-\nu(v)(t-s)} |h(s, x'_{k'} + (s - t'_{k'})v'_{k'}, v'')|$$

$$= C_{\varepsilon,T_0,N} \sum_j \int_{V_j} \int_{|v''|\le 3N} e^{-\nu(v)(t-s)} |h(s, y, v'')| \frac{1}{|\det\{\frac{\partial y}{\partial v'}\}|} dy dv'' ds ds_1$$

$$\le \frac{C_{\varepsilon,T_0,N}}{\delta} \int_0^t \int_0^{s_1} e^{-\nu_0 t} \int_{|v''|\le 3N} e^{\nu_0 s} \left\{ \int_\Omega h^2(s, y, v'') dy \right\}^{1/2} dv'' ds ds_1$$

$$\le C_{\varepsilon,T_0,N} \int_0^t ||f(s)|| ds,$$

where $f = \frac{h}{w}$. We therefore conclude, summing over k and k' and collecting terms

$$||h(t,x,v)\mathbf{1}_{A_\alpha}||_\infty \le \{1 + C_K t\}e^{-\nu_0 t}||h_0||_\infty + \{\frac{C}{N} + C_{N,T_0}\varepsilon\} \sup_s e^{-\frac{\nu_0}{2}\{t-s\}}||h(s)||_\infty$$

$$+C_{\varepsilon,N,T_0} \int_0^t ||f(s)|| ds. \tag{182}$$

STEP 2: Estimate of $h(t,x,v)$. We now further plug (182) back in: $h(t,x,v) = G(t,s)h_0 + \int_0^t G(t,s_1)\{K_w h(s_1)\} ds_1$ to get

$$||h(t)||_\infty \le e^{-\nu_0 t}||h_0||_\infty + \int_0^t e^{-\nu_0\{t-s_1\}}||K_w h||_\infty(s_1) ds. \tag{183}$$

But $\{K_w h\}(s_1, x, v) = \int K_w(v, v')h(s_1, x, v') dv'$ and we split it as

$$\int K_w(v, v')h(s_1, x, v')\{1 - \mathbf{1}_{A_\alpha(x,v')}\} dv' + \int K_w(v, v')h(s_1, x, v')\mathbf{1}_{A_\alpha(x,v')} dv'.$$

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By the definition of A_α in (179), the first term is bounded by

$$\left(\int_{|v'|\geq N, \text{ or } |v'|\leq \frac{1}{N}} |K_w(v,v')|dv' + \int_{\alpha(x,v')\leq \frac{1}{N}} |K_w(v,v')| \right) ||h(s_1)||_\infty.$$

By (45) and (124) if necessary, $\int_{|v'|\geq N, \text{ or } |v'|\leq \frac{1}{N}} |K_w(v,v')|dv' = o(1)$ as $N \to \infty$. From $\alpha(x,v') \leq \frac{1}{N}$, $\xi^2(x) + [v'\cdot \nabla\xi(x)]^2 \leq \frac{1}{N}$. For N large, $x \backsim \partial\Omega$ and $|\nabla\xi(x)| \geq \frac{1}{2}$ so that

$$\int_{\alpha(x,v')\leq \frac{1}{N}} |K_w(v,v')|dv' \leq \int_{|v'\cdot \frac{\nabla\xi(x)}{|\nabla\xi(x)|}|\leq \frac{2}{\sqrt{N}}} |K_w(v,v')|dv' = o(1)$$

as $N \to \infty$. We apply (182) to the second term to bound $||\{K_w h\}(s_1)||_\infty$ as

$$\{1+C_K s_1\}e^{-\nu_0 s_1}||h_0||_\infty + \{o(1) + \frac{C}{N} + C_{N,T_0}\varepsilon\} \sup_s e^{-\frac{\nu_0}{2}\{s_1-s\}}||h(s)||_\infty + C_{\varepsilon,N,T_0} \int_0^{s_1} ||f(s)||ds.$$

Hence, by (183), $||h(t)||_\infty$ is bounded by

$$e^{-\nu_0 t}||h_0||_\infty + \int_0^t e^{-\nu_0 t}\{1+C_K s_1\}||h_0||_\infty +$$

$$+ \int_0^t e^{-\nu_0\{t-s_1\}}\{o(1) + \frac{C}{N} + C_{N,T_0}\varepsilon\} \sup_{s\leq t} e^{-\frac{\nu_0}{2}\{s_1-s\}}||h(s)||_\infty + C_{\varepsilon,N,T_0} \int_0^{s_1} ||f(s)||ds\}ds_1$$

$$\leq \{1+C_K t^2\}e^{-\nu_0 t}||h_0||_\infty + C\{o(1) + \frac{1}{N} + C_{N,T_0}\varepsilon\} \sup_{s\leq t}\{e^{-\frac{\nu_0}{4}\{t-s\}}||h(s)||_\infty\} + C_{\varepsilon,N,T_0} \int_0^t ||f(s)||ds.$$

We choose T_0 large such that $2\{1 + C_K T_0^2\}e^{-\frac{\nu_0}{4}T_0} = e^{-\lambda T_0}$, for some $\lambda > 0$. We then further choose N large, and then ε sufficiently small such that $C\{o(1) + \frac{1}{N} + C_{N,T_0}\varepsilon\} < \frac{1}{2}$. We there have

$$\sup_{0\leq s\leq t} \{e^{\frac{\nu_0}{4}s}||h(s)||_\infty\} \leq 2\{1+C_K t^2\}||h_0||_\infty + C_{T_0} \int_0^t ||f(s)||ds.$$

Choosing $s = t = T_0$, we deduce the finite-time estimate (143), and our theorem follows from Lemma 29. ∎

4.4 L^∞ Decay of Diffuse Reflection

4.4.1 Infinite Cycles and L^∞ Bound for Diffuse $G(t)$

In this section, we study the L^∞ decay of the diffuse reflection. Define $h = fw$ to satisfy

$$\{\partial_t + v \cdot \nabla_x + \nu\}h = 0, \ h(t,x,v)|_{\gamma_-} = \frac{1}{\tilde{w}(v)} \int_{\mathcal{V}(x)} h(t,x,v')\tilde{w}(v')d\sigma \quad (184)$$

$$\text{where } \mathcal{V}(x) = \{v' \in \mathbf{R}^3 : v' \cdot n(x) > 0\}, \ \tilde{w}(v) \equiv \frac{1}{w(v)\sqrt{\mu(v)}}, \quad (185)$$

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and by (15), the probability measure $d\sigma = d\sigma(x)$ is given by

$$d\sigma(x) = c_\mu \mu(v')\{n(x) \cdot v'\}dv'. \tag{186}$$

For $\frac{1}{4} - \theta > 0$ and $\theta > \theta_0$, and for ρ sufficiently small,

$$\tilde{w} = \frac{e^{\{\frac{1}{4}-\theta\}|v|^2}}{(1+\rho|v|^2)^\beta} \geq 1, \quad \int_\mathcal{V} \tilde{w}^2 d\sigma \backsim c_\mu \int_\mathcal{V} e^{-2\theta|v|^2}\{n(x) \cdot v\}dv = \frac{1}{16\theta^2} < \frac{1}{16\theta_0^2}. \tag{187}$$

We have used the normalization (15) and a change of variable $v = \sqrt{\frac{1}{4\theta}}v'$ to evaluate the integral.

Definition 36 *Fix any point $(t, x, v) \notin \gamma_0$, and let $(t_0, x_0, v_0) = (t, x, v)$. Define the back-time cycle as*

$$(t_{k+1}, x_{k+1}, v_{k+1}) = (t_k - t_\mathbf{b}(x_k, v_k), x_\mathbf{b}(x_k, v_k), v_{k+1}), \ \ for \ v_{k+1} \in \mathcal{V}_{k+1} = \{v_{k+1} \cdot n(x_{k+1}) > 0\}. \tag{188}$$

And

$$X_\mathbf{cl}(s; t, x, v) = \sum_k \mathbf{1}_{[t_{k+1}, t_k)}(s)\{x_k + (s - t_k)v_k\}, \quad V_\mathbf{cl}(s; t, x, v) = \sum_k \mathbf{1}_{[t_{k+1}, t_k)}(s)v_k.$$

We define the iterated integral for $k \geq 2$

$$\int_{\Pi_{l=1}^{k-1}\mathcal{V}_l} \Pi_{l=1}^{k-1} d\sigma_l \equiv \int_{\mathcal{V}_1} \cdots \left\{ \int_{\mathcal{V}_{k-1}} d\sigma_{k-1} \right\} d\sigma_1 \tag{189}$$

We note that each v_l $(l = 1, 2, \ldots)$ are all independent variables, however, the phase space $\mathcal{V}_l$ implicitly depends on $(t, x, v, v_1, v_2, \ldots v_{l-1})$. We first show that the set in the phase space $\Pi_{l=1}^{k-1}\mathcal{V}_l$ not reaching $t = 0$ after k bounces is small when k is large.

Lemma 37 *For any $\varepsilon > 0$, there exists $k_0(\varepsilon, T_0)$ such that for $k \geq k_0$, for all $(t, x, v), 0 \leq t \leq T_0$, $x \in \bar{\Omega}$ and $v \in \mathbf{R}^3$,*

$$\int_{\Pi_{l=1}^{k-1}\mathcal{V}_l} \mathbf{1}_{\{t_k(t, x, v, v_1, v_2\ldots, v_{k-1}) > 0\}} \Pi_{l=1}^{k-1} d\sigma_l \leq \varepsilon.$$

Proof. Choosing $0 < \delta < 1$ sufficiently small, we further define non-grazing sets for $1 \leq l \leq k - 1$ as

$$\mathcal{V}_l^\delta = \{v_l \in \mathcal{V}_l : \ v_l \cdot n(x_l) \geq \delta\} \cap \{v_l \in \mathcal{V}_l : \ |v_l| \leq \frac{1}{\delta}\}.$$

Clearly, by the same argument in (87),

$$\int_{\mathcal{V}_l \backslash \mathcal{V}_l^\delta} d\sigma_l \leq \int_{v_l \cdot n(x_l) \leq \delta} d\sigma_l + \int_{|v_l| \geq \frac{1}{\delta}} d\sigma_l \leq C\delta, \tag{190}$$

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where C is independent of l. On the other hand, if $v_l \in \mathcal{V}_l^\delta$, then from diffusive back-time cycle (188), we have $x_l - x_{l+1} = (t_l - t_{l+1})v_l$. By (40) in Lemma 6, since $|v_l| \le \frac{1}{\delta}$, and $v_l \cdot n(x_l) \ge \delta$,

$$(t_l - t_{l+1}) \ge \frac{\delta^3}{C_\xi}.$$

Therefore, if $t_k(t, x, v, v_1, v_2..., v_{k-1}) > 0$, then there can be at most $\left[\frac{C_\xi T_0}{\delta^3}\right] + 1$ number of $v_l \in \mathcal{V}_l^\delta$ for $1 \le l \le k-1$. We therefore have

$$\int_{\mathcal{V}_1} \cdots \left\{ \int_{\mathcal{V}_{k-1}} \mathbf{1}_{\{t_k > 0\}} d\sigma_{k-1} \right\} d\sigma_{k-2}...d\sigma_1$$

$$\le \sum_{j=1}^{\left[\frac{C_\xi T_0}{\delta^3}\right]+1} \int_{\{\text{There are exactly } j \text{ of } v_{l_i} \in \mathcal{V}_{l_i}^\delta, \text{ and } k-1-j \text{ of } v_{l_i} \notin \mathcal{V}_{l_i}^\delta\}} \Pi_{l=1}^{k-1} d\sigma_l$$

$$\le \sum_{j=1}^{\left[\frac{C_\xi T_0}{\delta^3}\right]+1} \binom{k-1}{j} |\sup_l \int_{\mathcal{V}_l^\delta} d\sigma_l|^j \left\{ \sup_l \int_{\mathcal{V}_l \setminus \mathcal{V}_l^\delta} d\sigma_l \right\}^{k-j-1}.$$

Since $d\sigma$ is a probability measure, $\int_{\mathcal{V}_l^\delta} d\sigma_l \le 1$, and

$$\left\{ \int_{\mathcal{V}_l \setminus \mathcal{V}_l^\delta} d\sigma_l \right\}^{k-j-1} \le \left\{ \int_{\mathcal{V}_l \setminus \mathcal{V}_l^\delta} d\sigma_l \right\}^{k-2-\left[\frac{C_\xi T_0}{\delta^3}\right]} \le \{C\delta\}^{k-2-\left[\frac{C_\xi T_0}{\delta^3}\right]}.$$

But $\binom{k-1}{j} \le \{k-1\}^j \le \{k-1\}^{\left[\frac{C_\xi T_0}{\delta^3}\right]+1}$, we deduce that

$$\int \mathbf{1}_{\{t_k > 0\}} \Pi_{l=1}^{k-1} d\sigma_l \le \{k-1\}^{\left[\frac{C_\xi T_0}{\delta^3}\right]+1} \{C\delta\}^{k-2-\left[\frac{C_\xi T_0}{\delta^3}\right]}.$$

For $\varepsilon > 0$, our lemma follows for $C\delta < 1$, and $k >> \left[\frac{C_\xi T_0}{\delta^3}\right] + 1$. ∎

Lemma 38 *Assume that $h, \frac{q}{\nu} \in L^\infty$ satisfy $\{\partial_t + v \cdot \nabla_x + \nu\}h = q(t, x, v)$, with the diffuse boundary condition (184). Recall the diffusive cycles in (188). Then for any $0 \le s \le t$, for almost every x, v, if $t_1(t, x, v) \le s$,*

$$h(t, x, v) = e^{\nu(s-t)} h(s, x - v(t-s), v) + \int_s^t e^{\nu(\tau-t)} q(\tau, x - v(t-\tau), v) d\tau; \quad (191)$$

If $t_1(t, x, v) > s$, then for $k \ge 2$,

$$h(t, x, v) = \int_{t_1}^t e^{\nu(\tau-t)} q(\tau, x - v(t-\tau), v) d\tau + \frac{e^{\nu(v)(t_1-t)}}{\tilde{w}(v)} \int_{\Pi_{j=1}^{k-1} \mathcal{V}_j} H$$

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where H is given by

$$\sum_{l=1}^{k-1} \mathbf{1}_{\{t_l > s, t_{l+1} \le s\}} h(s, x_l + (s - t_l)v_l, v_l) d\Sigma_l(s)$$

$$+ \sum_{l=1}^{k-1} \int_s^{t_l} \mathbf{1}_{\{t_l > s, t_{l+1} \le s\}} q(\tau, x_l + (\tau - t_l)v_l, v_l) d\Sigma_l(\tau) d\tau$$

$$+ \sum_{l=1}^{k-1} \int_{t_{l+1}}^{t_l} \mathbf{1}_{\{t_{l+1} > s\}} q(\tau, x_l + (\tau - t_l)v_l, v_l) d\Sigma_l(\tau) d\tau$$

$$+ \mathbf{1}_{\{t_k > s\}} h(t_k, x_k, v_{k-1}) d\Sigma_{k-1}(t_k), \tag{192}$$

and $d\Sigma_{k-1}(t_k)$ is evaluated at $s = t_k$ of

$$d\Sigma_l(s) = \{\Pi_{j=l+1}^{k-1} d\sigma_j\} \{e^{\nu(v_l)(s-t_l)} \tilde{w}(v_l) d\sigma_l\} \Pi_{j=1}^{l-1} \{e^{\nu(v_j)(t_{j+1}-t_j)} d\sigma_j\}. \tag{193}$$

Proof. When $k = 2$, if $t_1(t, x, v) \le s$, then (191) is clearly valid. If $t_1(t, x, v) > s$,

$$h(t, x, v)\mathbf{1}_{\{t_1 > s\}} = e^{\nu(v)(t_1 - t)} h(t_1, x_1, v) + \int_{t_1}^t e^{\nu(v)(\tau - t)} q(\tau, x + (\tau - t)v, v) d\tau. \tag{194}$$

Since $\frac{d\{e^{\nu s} h\}}{ds} = e^{\nu s} q$ along a trajectory $\frac{dx}{dt} = v, \frac{dv}{dt} = 0$ almost everywhere, the first term can be expressed (almost everywhere) by the diffuse boundary condition (184) and part 4 of Lemma 6 as

$$\frac{e^{\nu(v)(t_1-t)}}{\tilde{w}(v)} \int_{\mathcal{V}_1} h(t_1, x_1, v_1) \tilde{w}(v_1) d\sigma_1$$

$$= \frac{e^{\nu(v)(t_1-t)}}{\tilde{w}(v)} \int_{\mathcal{V}_1} \mathbf{1}_{\{t_1 > s, t_2 \le s\}} e^{\nu(v_1)(s-t_1)} h(s, x_1 + v_1(s - t_1), v_1) \tilde{w}(v_1) d\sigma_1$$

$$+ \frac{e^{\nu(v)(t_1-t)}}{\tilde{w}(v)} \int_s^{t_1} \int_{\mathcal{V}_1} \mathbf{1}_{\{t_1 > s, t_2 \le s\}} e^{\nu(v_1)(\tau-t_1)} q(\tau, x_1 + v_1(\tau - t_1), v_1) \tilde{w}(v_1) d\sigma_1 d\tau$$

$$+ \frac{e^{\nu(v)(t_1-t)}}{\tilde{w}(v)} \int_{\mathcal{V}_1} \mathbf{1}_{\{t_2 > s\}} e^{\nu(v_1)\{t_2 - t_1\}} h(t_2, x_1 + v_1(t_2 - t_1), v_1) \tilde{w}(v_1) d\sigma_1$$

$$+ \frac{e^{\nu(v)(t_1-t)}}{\tilde{w}(v)} \int_{t_2}^{t_1} \int_{\mathcal{V}_1} \mathbf{1}_{\{t_2 > s\}} e^{\nu(v_1)(\tau-t_1)} q(\tau, x_1 + v_1(\tau - t_1), v_1) \tilde{w}(v_1) d\sigma_1 d\tau.$$

Therefore, the formula (192) is valid for $k = 2$. Assume that (192) is valid for $k \ge 2$, then for $k + 1$, we further split the last term in (192) with $t_k > 0$ into

$$h(t_k, x_k, v_{k-1}) \tilde{w}(v_{k-1}) = \int_{\mathcal{V}_k} h(t_k, x_k, v_k) \tilde{w}(v_k) d\sigma_k = \int_{\mathcal{V}_k} \mathbf{1}_{\{t_k > s, t_{k+1} \le s\}} + \mathbf{1}_{\{t_{k+1} > s\}}.$$

For the first term, we further integrate along the characteristics $\frac{dx}{dt} = v, \frac{dv}{dt} = 0$ to reach the plane $t = s$ as

$$\int_{\mathcal{V}_k} \mathbf{1}_{t_{k+1} \le s < t_k} \{e^{\nu(v_k)(s-t_k)} h(s, x_k + (s-t_k)v_k, v_k) + \int_s^{t_k} e^{\nu(v_k)(\tau-t_k)} q(\tau, x_k + (\tau - t_k)v_k, v_k) d\tau\} \tilde{w}(v_k) d\sigma_k;$$

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For the second term, we integrate along the characteristics $\frac{dx}{dt} = v$, $\frac{dv}{dt} = 0$ to $t = t_{k+1} > s$ to get

$$\int_{\mathcal{V}_k} \mathbf{1}_{t_{k+1}>s}\{e^{\nu(v_k)(t_{k+1}-t_k)}h(t_{k+1}, x_{k+1}, v_k) + \int_{t_{k+1}}^{t_k} e^{\nu(v_k)(\tau-t_k)}q(\tau, x_k+(\tau-t_k)v_k, v_k)d\tau\}\tilde{w}(v_k)d\sigma_k.$$

We then deduce our lemma by adding $\int_{\mathcal{V}_k} d\sigma_k = 1$ inside the rest of the terms so that all the integrations are over $\Pi_{l=1}^{k}\mathcal{V}_l$ instead of $\Pi_{l=1}^{k-1}\mathcal{V}_l$. ∎

Lemma 39 *Let $h_0 \in L^\infty$ and assume (21). There exits a unique solution $h(t) = G(t)h_0 \in L^\infty$ to (29) with the diffuse boundary condition (184) and*

$$\sup_{0\leq t\leq 1} \{e^{\nu_0 t}||\{G(t)h_0\}\mathbf{1}_{t_1>0}||_\infty\} \leq e^{\frac{\nu_0}{2}}||\frac{h_0}{\tilde{w}}||_\infty, \quad ||\{G(t)h_0\}\mathbf{1}_{t_1\leq 0}||_\infty \leq ||e^{-t\nu(v)}h_0||_\infty$$

(195)

In particular,

$$\sup_{t\geq 1} e^{\frac{\nu_0}{2}t}||G(t)h_0||_\infty \leq e^{\nu_0}\max\{||\frac{h_0}{\tilde{w}}||_\infty, ||e^{-\nu(v)+\nu_0}h_0||_\infty\}. \quad (196)$$

Proof. Given any $m \geq 1$, we first construct a solution to $\{\partial_t + v\cdot\nabla_x + \nu\}h^m = 0$, with the following approximate boundary and initial conditions as

$$h^m(t, x, v) = \left\{1 - \frac{1}{m}\right\}\frac{1}{\tilde{w}(v)}\int_{\mathcal{V}} h^m(t, x, v')\tilde{w}(v')d\sigma(x), \quad (197)$$

$$h^m(0, x, v) = h_0\mathbf{1}_{\{|v|\leq m\}}.$$

Then $\tilde{h}^m \equiv h^m\tilde{w}$ satisfies $\{\partial_t + v\cdot\nabla_x + \nu\}\tilde{h}^m = 0$ but with

$$\tilde{h}^m(t, x, v) = \left\{1 - \frac{1}{m}\right\}\int_{\mathcal{V}} \tilde{h}^m(t, x, v')d\sigma(x). \quad (198)$$

Clearly, since $\int d\sigma = 1$, this boundary operator maps L^∞ to L^∞ with norm bounded by $1 - \frac{1}{m}$, and initially

$$||\tilde{h}^m(0)||_\infty = \sup |h^m(0, x, v)\tilde{w}| = ||h_0\mathbf{1}_{\{|v|\leq m\}}\tilde{w}||_\infty \leq C_{m,\theta}||h_0||_\infty < \infty.$$

Therefore, by Lemma 23, there exists a solution $\tilde{h}^m(t, x, v) \in L^\infty$ to (29) with (198), so that we have constructed $h^m = \frac{\tilde{h}^m}{\tilde{w}}$ with (197), which obviously is bounded. Such a solution is unique by the transformation $f^m = \frac{h^m}{w} \in L^2$ with $\int_0^t ||f^m(s)||_\gamma^2 ds < \infty$.

In order to take $m \to \infty$ in (197), we need to obtain an uniform L^∞ bound (195) and (196) for h^m, which is more delicate. We first claim that it suffices to show (195) to derive (196) for h^m. Letting $t = 1$ in two parts of (195), since $\tilde{w} \geq 1$ from (187) and $e^{-\frac{\nu_0}{2}}e^{\nu_0} \geq 1$, we have

$$||h^m(1)||_\infty \leq e^{-\frac{\nu_0}{2}}\max\{||h(0)||_\infty, ||e^{-\nu(v)+\nu_0}h(0)||_\infty\} \leq e^{-\frac{\nu_0}{2}}||h(0)||_\infty. \quad (199)$$

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For any $l \le t < l+1$, we can repeatedly apply (199) to get:

$$||h^m(l)||_\infty \le e^{-\frac{\nu_0}{2}}||h^m(l-1)||_\infty.$$

Therefore, by (195), we deduce (196) as

$$
\begin{aligned}
||h^m(t)||_\infty &\le e^{\frac{\nu_0}{2}}||h^m(l)||_\infty \le ||h^m(l-1)||_\infty \\
&\le e^{-\frac{\nu_0(l-2)}{2}}||h^m(1)||_\infty \\
&\le e^{-\frac{\nu_0(l-1)}{2}} \max\{||\frac{h(0)}{\tilde{w}}||_\infty, ||e^{-\nu(v)+\nu_0}h(0)||_\infty\} \\
&\le e^{\nu_0}e^{-\frac{\nu_0}{2}t} \max\{||\frac{h(0)}{\tilde{w}}||_\infty, ||e^{-\nu(v)+\nu_0}h(0)||_\infty\}.
\end{aligned}
$$

The rest of the proof is devoted to the validity of (195). If $t_1(t,x,v) \le 0$, we have $\{G(t)h_0^m\}(t,x,v) = e^{-\nu(v)t}h_0^m(x-tv,v)$ and (195) is clearly valid.

We now consider $t_1(t,x,v) > 0$, then the back-time trajectory first hits the boundary. As in the proof of (192) with $q \equiv 0$, we can ignore powers of the factor $1 - \frac{1}{m}$ to bound $|h^m(t,x,v)|$ by $\frac{e^{\nu(v)(t_1-t)}}{\tilde{w}}\times$

$$
\begin{aligned}
&\sum_{l=1}^{k-1}\int_{\Pi_{l=1}^{k-1}\mathcal{V}_l} \mathbf{1}_{\{t_l>0, t_{l+1}\le 0\}}|h^m(0, x_l-t_lv_l, v_l)|d\Sigma_l(0) \\
&+ \int_{\Pi_{l=1}^{k-1}\mathcal{V}_l} \mathbf{1}_{\{t_k>0\}}|h^m(t_k, x_k, v_{k-1})|d\Sigma_{k-1}(t_k). \quad\quad (200)
\end{aligned}
$$

Over the second small set $\mathbf{1}_{\{t_k>0\}}$, choose any $\varepsilon(\nu_0) > 0$ such that

$$(1 - 2\sqrt{\varepsilon})e^{\frac{\nu_0}{2}} > 1, \quad\quad (201)$$

then choose $k_0(\varepsilon)$ by Lemma 37 with $T_0 = 1$. By Lemma 37 with $T_0 = 1$, for $k = k_0(\varepsilon)$, $\int_{\Pi_{l=1}^{k-1}\mathcal{V}_l} \mathbf{1}_{\{t_k>0\}}\Pi_{l=1}^{k-1}d\sigma_l < \varepsilon$. We further split the second integral in (200) into $\{t_k > 0, t_{k+1} \le 0\}$ and $\{t_{k+1} > 0\}$ in $\mathcal{V}_k$ with $d\sigma_k$. Integrating along the characteristic for the first part $\{t_k > 0, t_{k+1} \le 0\}$ yields: $\frac{e^{\nu(v)(t_1-t)}}{\tilde{w}}\times$

$$
\begin{aligned}
&\int_{\Pi_{l=1}^{k}\mathcal{V}_l} \mathbf{1}_{\{t_k>0, t_{k+1}\le 0\}}|h^m(0, x_k-v_kt_k, v_k)|d\Sigma_k(0) \\
&+ \int_{\Pi_{l=1}^{k}\mathcal{V}_l} \mathbf{1}_{\{t_{k+1}>0\}}|h^m(t_k, x_k, v_{k-1})|d\Sigma_{k-1}(t_k)d\sigma_k. \quad\quad (202)
\end{aligned}
$$

Since $t_1(t_k, x_k, v_k) > 0$ over the set $\{t_{k+1} > 0\}$, we deduce that

$$\mathbf{1}_{\{t_{k+1}>0\}}|h^m(t_k, x_k, v_{k-1})| \le \sup_{x,v}|h^m(t_k, x, v)\mathbf{1}_{\{t_1>0\}}|.$$

From (193), the exponential in $d\Sigma_l(s)$ is bounded by $e^{\nu_0(s-t_1)}$. Since $\int_{\mathcal{V}_k} d\sigma_k = 1$, by Lemma 37, the last part in (202) is then bounded by

$$\frac{e^{\nu_0(t_k-t)}}{\tilde{w}}||h^m(t_k)\mathbf{1}_{t_1>0}||_\infty \int_{\Pi_{l=1}^{k-1}\mathcal{V}_l} \mathbf{1}_{\{t_k>0\}}\tilde{w}(v_{k-1})\Pi_{l=1}^{k-1}d\sigma_l \tag{203}$$

$$\leq \sup_{0\leq s\leq t\leq 1}\{e^{\nu_0(s-t)}||h^m(s)\mathbf{1}_{t_1>0}||_\infty\}\left\{\int_{\Pi_{l=1}^{k-1}\mathcal{V}_l}\mathbf{1}_{\{t_k>0\}}\Pi_{l=1}^{k-1}d\sigma_l\right\}^{1/2}\left\{\int\tilde{w}^2(v_{k-1})d\sigma_{k-1}\right\}^{1/2}$$

$$\leq 2\sqrt{\varepsilon}\sup_{0\leq s\leq t\leq 1}\{e^{\nu_0(s-t)}||h^m(s)\mathbf{1}_{t_1>0}||_\infty\},$$

We have used (187) for $\theta < \theta_0 \backsim \frac{1}{4}$.

On the other hand, inserting $\int_{\mathcal{V}_k} d\sigma_k = 1$ into the main contribution in (200), and combining with the first term in (202) yields:

$$\frac{e^{\nu(v)(t_1-t)}}{\tilde{w}}\int_{\Pi_{l=1}^k\mathcal{V}_l}\sum_{l=1}^k\mathbf{1}_{\{t_l>0,t_{l+1}\leq 0\}}|h^m(0,x_l-t_lv_l,v_l)|$$

$$\times\{\Pi_{j=l+1}^k d\sigma_j\}\{\tilde{w}(v_l)e^{-\nu(v_l)t_l}d\sigma_l\}\Pi_{j=1}^{l-1}\{e^{\nu(v_j)(t_{j+1}-t_j)}d\sigma_j\}$$

$$\leq \frac{e^{-\nu_0 t}}{\tilde{w}}||\frac{h_0^m}{\tilde{w}}||_\infty\int\sum_{l=1}^k\mathbf{1}_{\{t_l>0,t_{l+1}\leq 0\}}\Pi_{j=1}^k\tilde{w}^2(v_j)d\sigma_j,$$

since $\tilde{w}(v_j) \geq 1$ by (21) and (187). Note $\sum_{l=1}^k\mathbf{1}_{\{t_l>0,t_{l+1}\leq 0\}} = \mathbf{1}_{\{t_{k+1}\leq 0\}}$. By (201), we can further choose θ_0 near $1/4$ such that $\left(\frac{1}{4\theta_0}\right)^{2k_0(\varepsilon)} \leq (1-2\sqrt{\varepsilon})e^{\frac{\nu_0}{2}}$ in (187). We therefore get

$$\int_{\Pi_{l=1}^k\mathcal{V}_l}\sum_{l=1}^k\mathbf{1}_{\{t_l>0,t_{l+1}\leq 0\}}\Pi_{l=1}^k\tilde{w}^2(v_l)d\sigma_l$$

$$= \int_{\Pi_{l=1}^k\mathcal{V}_l}\mathbf{1}_{\{t_{k+1}\leq 0\}}\Pi_{l=1}^k\tilde{w}^2(v_l)d\sigma_l$$

$$\leq \Pi_{l=1}^k\left\{\int_{\mathcal{V}_l}\tilde{w}^2(v_l)d\sigma_l\right\} \leq (1-2\sqrt{\varepsilon})e^{\frac{\nu_0}{2}}.$$

for $k = k_0(\varepsilon)$. Hence, combining with (203), we have

$$\sup_{x,v}\{e^{\nu_0 t}|h_m(t,x,v)\mathbf{1}_{\{t_1>0\}}|\} \leq 2\sqrt{\varepsilon}\sup_{0\leq s\leq 1}\{e^{\nu_0 s}||h^m(s)\mathbf{1}_{t_1>0}||_\infty\}+(1-2\sqrt{\varepsilon})e^{\frac{\nu_0}{2}}||\frac{h_0^m}{\tilde{w}}||_\infty.$$

Taking $\sup_{0\leq t\leq 1}$ and absorbing the first term on the right hand side, we obtain

$$\sup_{0\leq t\leq 1}e^{\nu_0 t}||h_m(t)\mathbf{1}_{\{t_1>0\}}||_\infty \leq e^{\frac{\nu_0}{2}}||\frac{h_0^m}{\tilde{w}}||_\infty.$$

Letting $t = 1$, and we deduce (195) uniform in m. We then deduce our lemma by letting $m \to \infty$. $\blacksquare$

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4.4.2 Continuity of Diffuse $G(t)$.

Lemma 40 *Let Ω be strictly convex as in (4) and (21) be valid. Let h and q satisfy $\{\partial_t + v \cdot \nabla_x + \nu\}h = q(t, x, v)$, with the diffuse boundary condition (184). Assume $h(0, x, v) = h_0(x, v)$, continuous for $(x, v) \notin \gamma_0$, and $q(t, x, v)$ is continuous in the interior of $[0, \infty] \times \Omega \times \mathbf{R}^3$ with $\sup_{[0, \infty] \times \Omega \times \mathbf{R}^3} |\frac{q(t, x, v)}{\nu(v)}| < \infty$. Assume*

$$h_0(x, v)|_{\gamma_-} = \frac{1}{\tilde{w}(v)} \int_{\mathcal{V}} h_0(x, v')\tilde{w}(v')d\sigma. \tag{204}$$

Then for any t, $(x, v) \notin \gamma_0$, $h(t, x, v)$ is continuous.

Proof. We fix (t, x, v) such that $(x, v) \notin \gamma_0$, for any fixed k, we recall (192) with $s = 0$ for the expression of $h(t, x, v)$. Now we take any point $(\bar{t}, \bar{x}, \bar{v})$ near (t, x, v) and evaluate $h(\bar{t}, \bar{x}, \bar{v})$ by (192) with the same number of bounces (k−bounces), and with corresponding $\bar{t}_l, \bar{x}_l$ and $\bar{\mathcal{V}}_l$ and $d\bar{\sigma}_l$.

    Step 1. Reduction to the approximate of phase spaces. Since $\nu(v) \backsim |v|$ for large v and $\frac{1}{w}$ decays exponentially, by Lemma 39 and the Duhamel principle,

$$||h(t)||_\infty \leq ||G(t)h_0||_\infty + \int_0^t ||G(t-s)q(s)||_\infty ds \leq C(t, ||h_0||_\infty, \sup_{[0, \infty] \times \Omega \times \mathbf{R}^3} |\frac{q}{\nu}|), \tag{205}$$

For any $\varepsilon > 0$, by Lemma 37 and (205) , we can fix $k(\varepsilon, t)$ sufficiently large, such that the last terms in (192) for both $h(t, x, v)$ and $h(\bar{t}, \bar{x}, \bar{v})$ are bounded by

$$\{||h(t_k)||_\infty + ||h(\bar{t}_k)||\} \int_{\Pi_{l=1}^{k-1}\mathcal{V}_l} \mathbf{1}_{\{t_k > 0\}}\Pi_{l=1}^{k-1}d\sigma_l + \int_{\Pi_{l=1}^{k-1}\bar{\mathcal{V}}_l} \mathbf{1}_{\{\bar{t}_k > 0\}}\Pi_{l=1}^{k-1}d\bar{\sigma}_l \leq \frac{\varepsilon}{2}.$$

For the remaining sets $\mathbf{1}_{\{t_k \leq 0\}}$ and $\mathbf{1}_{\{\bar{t}_k \leq 0\}}$, for $\varepsilon_1 << \varepsilon$, define the non-grazing sets as

$$\mathcal{V}_l^{\varepsilon_1} = \{v_l : v_l \cdot n(x_l) \geq \varepsilon_1 \text{ and } |v_l| \leq \frac{1}{\varepsilon_1}\},$$

$$\bar{\mathcal{V}}_l^{\varepsilon_1} = \{v_l : v_l \cdot n(\bar{x}_l) \geq \varepsilon_1 \text{ and } |v_l| \leq \frac{1}{\varepsilon_1}\}.$$

We further split the integration region in (192) as

$$\int \mathbf{1}_{\{t_k \leq 0\}} = \int_{\{\text{there exists a } v_l \in \mathcal{V}_l \backslash \mathcal{V}_l^{\varepsilon_1}\}} \mathbf{1}_{\{t_k \leq 0\}} + \int_{\{\text{all } v_l \in \mathcal{V}_l^{\varepsilon_1}\}} \mathbf{1}_{\{t_k \leq 0\}};$$

$$\int \mathbf{1}_{\{\bar{t}_k \leq 0\}} = \int_{\{\text{there exists a } v_l \in \bar{\mathcal{V}}_l \backslash \bar{\mathcal{V}}_l^{\varepsilon_1}\}} \mathbf{1}_{\{\bar{t}_k \leq 0\}} + \int_{\{\text{all } v_l \in \bar{\mathcal{V}}_l^{\varepsilon_1}\}} \mathbf{1}_{\{\bar{t}_k \leq 0\}}.$$

Clearly, by (190), $\int_{\mathcal{V}_l \backslash \mathcal{V}_l^{\varepsilon_1}} d\sigma_l + \int_{\bar{\mathcal{V}}_l \backslash \bar{\mathcal{V}}_l^{\varepsilon_1}} d\bar{\sigma}_l \leq C\varepsilon_1$, so that from the boundedness of h_0 and $\frac{q}{\nu}$, the integrals in (192) over the almost grazing sets are small:

$$\left|\int_{\{\text{there exists } v_l \in \mathcal{V}_l \backslash \mathcal{V}_l^{\varepsilon_1}\}} \mathbf{1}_{\{t_k \leq 0\}}...\right| + \left|\int_{\{\text{there exists } v_l \in \bar{\mathcal{V}}_l \backslash \bar{\mathcal{V}}_l^{\varepsilon_1}\}} \mathbf{1}_{\{\bar{t}_k \leq 0\}}...\right| \leq C(h_0, \frac{q}{\nu}, k)\varepsilon_1 \leq \frac{\varepsilon}{4}.$$

Therefore, it suffices to compare only the integrations over the non-grazing sets $\Pi_{l=1}^{k-1}\mathcal{V}_l^{\varepsilon_1}\cap\{t_k\leq 0\}$ and $\Pi_{l=1}^{k-1}\bar{\mathcal{V}}_l^{\varepsilon_1}\cap\{\bar{t}_k\leq 0\}$. For any $1\leq l\leq k-1$, recalling the back-time diffuse cycle (188), if $\alpha(t_l)=\{v_l\cdot n(x_l)\}^2\geq\frac{\varepsilon_1}{2}>0$, then from convexity and the Velocity Lemma 5, we deduce

$$\alpha(t_{l+1})=\{v_l\cdot n(x_{l+1})\}^2\geq C\alpha(t_l)\geq C\frac{\varepsilon_1}{2}>0.$$

Hence x_{l+1},t_{l+1} are smooth functions of x_l and v_l from Lemma 6. A simple induction for l implies that x_l,t_l are smooth functions of $(v_1,...v_{l-1})\in\Pi_{j=1}^{l-1}\mathcal{V}_j^{\frac{\varepsilon_1}{2}}$:

$$|t_l-\bar{t}_l|+|x_l-\bar{x}_l|\rightarrow 0 \tag{206}$$

as $(\bar{t},\bar{x},\bar{v})\rightarrow(t,x,v)$, uniformly in $\Pi_{j=1}^{l-1}\mathcal{V}_j^{\frac{\varepsilon_1}{2}}$. Clearly $\Pi_{l=1}^{k-1}\mathcal{V}_l^{\varepsilon_1}\subset\Pi_{l=1}^{k-1}\mathcal{V}_l^{\frac{\varepsilon_1}{2}}$. Since $\bar{x}_1\backsim x_1$ by the Velocity Lemma 5, $\bar{\mathcal{V}}_1^{\varepsilon_1}\subset\mathcal{V}_1^{\frac{\varepsilon_1}{2}}$. A simple induction leads to (206) and $\Pi_{j=1}^{l-1}\bar{\mathcal{V}}_j^{\varepsilon_1}\subset\Pi_{j=1}^{l-1}\mathcal{V}_j^{\frac{\varepsilon_1}{2}}$ for $1\leq l\leq k-1$. Therefore, (206) is valid on both $\Pi_{l=1}^{k-1}\mathcal{V}_l^{\varepsilon_1}$ and $\Pi_{l=1}^{k-1}\bar{\mathcal{V}}_l^{\varepsilon_1}$, subsets of $\Pi_{l=1}^{k-1}\mathcal{V}_l^{\frac{\varepsilon_1}{2}}$.

Moreover, we have

$$\mathcal{V}_l^{\varepsilon_1}\setminus\bar{\mathcal{V}}_l^{\varepsilon_1} \equiv \{v_l\cdot n(x_l)\geq\varepsilon_1, v_l\cdot n(\bar{x}_l)<\varepsilon_1, \text{ and } |v_l|\leq\frac{1}{\varepsilon_1}\}.$$

$$\bar{\mathcal{V}}_l^{\varepsilon_1}\setminus\mathcal{V}_l^{\varepsilon_1} \equiv \{v_l\cdot n(x_l)<\varepsilon_1, v_l\cdot n(\bar{x}_l)\geq\varepsilon_1, \text{ and } |v_l|\leq\frac{1}{\varepsilon_1}\}.$$

By continuity (206), for $(\bar{t},\bar{x},\bar{v})\rightarrow(t,x,v)$, $x_l\rightarrow\bar{x}_l$, and both sets are contained in

$$\{\varepsilon_1-C|x_l-\bar{x}_l|\leq v_l\cdot n(x_l)\leq\varepsilon_1+C|x_l-\bar{x}_l|, \text{ and } |v_l|\leq\frac{1}{\varepsilon_1}\}$$

which have measure $\frac{C|x_l-\bar{x}_l|}{\varepsilon_1^2}$. We now define the approximate phase-spaces as:

$$B=\Pi_{l=1}^{k-1}[\mathcal{V}_l^{\varepsilon_1}\cap\bar{\mathcal{V}}_l^{\varepsilon_1}]\cap\{t_k\leq 0,\bar{t}_k\leq 0\}. \tag{207}$$

To estimate $\Pi_{l=1}^{k-1}\mathcal{V}_l^{\varepsilon_1}\setminus B$, by an induction on k, we get

$$|\Pi_{l=1}^{k-1}\mathcal{V}_l^{\varepsilon_1}\setminus\Pi_{l=1}^{k-1}[\mathcal{V}_l^{\varepsilon_1}\cap\bar{\mathcal{V}}_l^{\varepsilon_1}]|+|\Pi_{l=1}^{k-1}\mathcal{V}_l^{\varepsilon_1}\setminus\Pi_{l=1}^{k-1}\bar{\mathcal{V}}_l^{\varepsilon_1}|\leq C(\varepsilon_1,k)\sup_{1\leq l\leq k-1}|x_l-\bar{x}_l|.$$

Notice that $\Pi_{l=1}^{k-1}\mathcal{V}_l^{\varepsilon_1}\subset[\Pi_{l=1}^{k-1}\mathcal{V}_l^{\varepsilon_1}\setminus\Pi_{l=1}^{k-1}\bar{\mathcal{V}}_l^{\varepsilon_1}]\cup\Pi_{l=1}^{k-1}\bar{\mathcal{V}}_l^{\varepsilon_1}$, $\Pi_{l=1}^{k-1}\mathcal{V}_l^{\varepsilon_1}\setminus B$ is contained in

$$\Pi_{l=1}^{k-1}\mathcal{V}_l^{\varepsilon_1}\setminus\Pi_{l=1}^{k-1}[\mathcal{V}_l^{\varepsilon_1}\cap\bar{\mathcal{V}}_l^{\varepsilon_1}]\cup[\Pi_{l=1}^{k-1}\mathcal{V}_l^{\varepsilon_1}\cap\{t_k > 0\}]$$
$$\cup[\Pi_{l=1}^{k-1}\bar{\mathcal{V}}_l^{\varepsilon_1}\cap\{\bar{t}_k > 0\}]\cup[\Pi_{l=1}^{k-1}\mathcal{V}_l^{\varepsilon_1}\setminus\Pi_{l=1}^{k-1}\bar{\mathcal{V}}_l^{\varepsilon_1}]$$

From Lemma 37, both $\int_{\Pi_{l=1}^{k-1}\mathcal{V}_l\cap\{t_k>0\}}\Pi_{l=1}^{k-1}d\sigma_l$ and $\int_{\Pi_{l=1}^{k-1}\bar{\mathcal{V}}_l\cap\{\bar{t}_k>0\}}\Pi_{l=1}^{k-1}d\bar{\sigma}_l$ are bounded by $C\varepsilon$. By similar splitting for the set $\Pi_{l=1}^{k-1}\bar{\mathcal{V}}_l^{\varepsilon_1}\setminus B$, as $|x-\bar{x}|+|t-\bar{t}|+|v-\bar{v}|\rightarrow 0$, we deduce

$$\int_{\Pi_{l=1}^{k-1}\mathcal{V}_l^{\varepsilon_1}\setminus B}\Pi_{l=1}^{k-1}d\sigma_l+\int_{\Pi_{l=1}^{k-1}\bar{\mathcal{V}}_l^{\varepsilon_1}\setminus B}\Pi_{l=1}^{k-1}d\bar{\sigma}_l<4C\varepsilon+C(\varepsilon_1,k)\sup_{1\leq l\leq k-1}|x_l-\bar{x}_l|<5C\varepsilon.$$

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By L^2 bound for $\tilde{w}$ in (187) and Cauchy-Schwarz's inequality:

$$\int_{\Pi_{l=1}^{k-1}\mathcal{V}_l^{\varepsilon_1}\backslash B} \Pi_{l=1}^{k-1}d\Sigma_l(s) + \int_{\Pi_{l=1}^{k-1}\bar{\mathcal{V}}_l^{\varepsilon_1}\backslash B} \Pi_{l=1}^{k-1}d\bar{\Sigma}_l(s) \leq C(t)\sqrt{\varepsilon}.$$

Thanks to the boundedness of h_0 and $\frac{q}{\nu}$, to prove the continuity, it suffices to estimate the difference of $h(t,x,v)$ and $h(\bar{t},\bar{x},\bar{v})$ in (192), where the integrations are over the same set B.

Step 2. Continuity of $h(t,x,v)$ over B.

Case 1: $t_1(t,x,v) \leq 0$. In the case $t_1 < 0$, then $\bar{t}_1 < 0$ by continuity over the set B in (207). Then both $h(t,x,v)$ and $h(\bar{t},\bar{x},\bar{v})$ are given by the same formula (191). The continuity now follows from $(\bar{t},\bar{x},\bar{v}) \to (t,x,v)$ and the continuity of h_0 and q. Same argument also applies to the situation $t_1 = 0$ and $t_1 \leq 0$.

We only need to study the case $t_1 = 0$ but $\bar{t}_1 > 0$ in which $h(\bar{t},\bar{x},\bar{v})$ are given by the different expression (192). Over the set B, since $|\bar{v}_1 \cdot n(\bar{x}_1)| \geq \varepsilon_1 > 0$, from (40) that $\bar{t}_1 - \bar{t}_2 \geq \frac{\varepsilon_1^3}{C_\xi}$. But $\bar{t}_1 \to t_1 = 0$, we therefore deduce that $\bar{t}_2 < 0$. This implies for k large

$$B = \{\bar{t}_1 > 0, \bar{t}_2 \leq 0\} \cap B = \Pi_{l=1}^{k-1}\mathcal{V}_l^{\varepsilon_1} \cap \bar{\mathcal{V}}_l^{\varepsilon_1}.$$

Applying (192) to $h(\bar{t},\bar{x},\bar{v})$ over the set B with $\bar{t}_2 < 0$, by Step 1, we obtain

$$
\begin{aligned}
h(\bar{t},\bar{x},\bar{v}) \quad \sim \quad & \int_{\bar{t}_1}^{\bar{t}} e^{\nu(\tau-\bar{t})} q(\tau, \bar{x} - \bar{v}(\bar{t}-\tau), \bar{v}) d\tau + \\
& + \frac{e^{\nu(\bar{v})(\bar{t}_1-\bar{t})}}{\tilde{w}(\bar{v})} \int_{\mathcal{V}_1^{\varepsilon_1} \cap \bar{\mathcal{V}}_1^{\varepsilon_1}} \mathbf{1}_{\{\bar{t}_1>0,\bar{t}_2\leq 0\}} h_0(\bar{x}_1 - \bar{t}_1 v_1, v_1) \tilde{w}(v_1) e^{-\nu(v_1)\bar{t}_1} d\bar{\sigma}_1 \qquad (208) \\
& + \frac{e^{\nu(\bar{v})(\bar{t}_1-\bar{t})}}{\tilde{w}(\bar{v})} \int_{\mathcal{V}_1^{\varepsilon_1} \cap \bar{\mathcal{V}}_1^{\varepsilon_1}} \int_0^{\bar{t}_1} \mathbf{1}_{\{\bar{t}_1>0,\bar{t}_2\leq 0\}} q(\bar{x}_1 + (\tau-\bar{t}_1)v_1, v_1) \tilde{w}(v_1) e^{\nu(v_1)(\tau-\bar{t}_1)} d\bar{\sigma}_1 d\tau.
\end{aligned}
$$

Since $\bar{t}_1 \to t_1 = 0$, it follows the last term above is small from the boundedness of $\frac{q}{\nu}$. The first term on the right hand side of (208) tends to

$$\int_0^t e^{\nu(\tau-t)} q(\tau, x - v(t-\tau), v) d\tau,$$

as second part of $h(t,x,v)$ in (191), from the continuity of q. Since $\mathbf{1}_{\{\bar{t}_1>0,\bar{t}_2\leq 0\}} \equiv 1$ over $\Pi_{l=1}^{k-1}\mathcal{V}_l^{\varepsilon_1} \cap \bar{\mathcal{V}}_l^{\varepsilon_1}$ in this case, and by $\bar{t}_1 \curvearrowright 0$, $\bar{x}_1 \curvearrowright x_1 \in \partial\Omega$, the second term on the right hand side of (208) tends to

$$\frac{e^{-\nu(v)t}}{\tilde{w}(v)} \int_{\mathcal{V}_1^{\varepsilon_1}} h_0(x_1, v_1) \tilde{w}(v_1) d\sigma_1 \curvearrowright e^{-\nu(v)t} h_0(x_1, v) \sim e^{-\nu(v)t} h_0(x - tv, v),$$

by the continuity of h_0 away from γ_0 and the compatibility condition (204). Therefore, we have shown $h(\bar{t},\bar{x},\bar{v}) \to h(t,x,v)$ by (191).

CASE 2: $t_1(t, x, v) > 0$. From continuity, $\bar{t}_1 > 0$ and

$$\int_{t_1}^t e^{\nu(\tau - t)} q(\tau, x - v(t - \tau), v) d\tau \curvearrowright \int_{\bar{t}_1}^{\bar{t}} e^{\nu(\tau - \bar{t})} q(\tau, \bar{x} - v(\bar{t} - \tau), \bar{v}) d\tau.$$

It thus sufficient to only study integrals (192) over B for both $h(t, x, v)$ and $h(\bar{t}, \bar{x}, \bar{v})$. We further split

$$B = \sum_{i,m} B \cap \{t_{i+1} \le 0, t_i > 0; \bar{t}_{m+1} \le 0, \bar{t}_m > 0\} \equiv \sum_{i,m} B_{im},$$

and $h(t, x, v) - h(\bar{t}, \bar{x}, \bar{v}) \curvearrowright \sum_{i,m} \int_{B_{im}}$. It suffices to estimate the difference over each B_{im}, which can be rewritten from (192) as:

$$
\begin{aligned}
&\frac{e^{\nu(v)(t_1 - t)}}{\tilde{w}(v)} \times \{ \int_{B_{im}} h_0(x_i - t_i v_i, v_i) d\Sigma_i(0) \\
&+ \int_{B_{im}} \sum_{j=1}^{i-1} \int_{t_{j+1}}^{t_j} q(\tau, x_j + (\tau - t_j) v_j, v_j) d\Sigma_j(\tau) d\tau \\
&+ \int_0^{t_i} \int_{B_{im}} q(\tau, x_i + (\tau - t_i) v_i, v_i) d\Sigma_i(\tau) d\tau \} \qquad\qquad (209) \\
&- \frac{e^{\nu(\bar{v})(\bar{t}_1 - \bar{t})}}{\tilde{w}(\bar{v})} \times \{ \int_{B_{im}} h_0(\bar{x}_m - \bar{t}_m v_m, v_m) d\Sigma_m(s) \\
&+ \int_{B_{im}} \sum_{j=1}^{m-1} \int_{\bar{t}_{j+1}}^{\bar{t}_j} q(\tau, \bar{x}_j + (\tau - \bar{t}_j) v_j, v_j) d\bar{\Sigma}_j(\tau) d\tau \\
&+ \int_0^{\bar{t}_m} \int_{B_{im}} q(\tau, \bar{x}_m + (\tau - \bar{t}_m) v_m, v_m) d\bar{\Sigma}_m(\tau) d\tau \}.
\end{aligned}
$$

By (40) in Lemma 6, $t_i - t_{i+1} \ge \frac{\varepsilon_1^3}{C_\xi} > 0$. For $\varepsilon_2 << \varepsilon_1$, we further split

$$\{t_i > 0, t_{i+1} \le 0\} = \{t_i > \varepsilon_2, t_{i+1} \le -\varepsilon_2\} \cup \{0 \le t_i \le \varepsilon_2, t_{i+1} \le 0\} \cup \{t_i > \varepsilon_2, -\varepsilon_2 < t_{i+1} \le 0\}.$$

CASE 2a: On the set $B_{im} \cap \{t_i > \varepsilon_2, t_{i+1} \le -\varepsilon_2\}$. By continuity (206), for $(\bar{t}, \bar{x}, \bar{v}) \to (t, x, v)$,

$$\bar{t}_i > \frac{\varepsilon_2}{2}, \bar{t}_{i+1} \le -\frac{\varepsilon_2}{2},$$

then $h(\bar{t}, \bar{x}, \bar{v})$ has the same expression as $h(t, x, v)$ with $m = i$ in (209), and the difference in (209) is small over this set, from the continuity of h_0 and q.

CASE 2b: On the set $B_{im} \cap \{0 < t_i \le \varepsilon_2, t_{i+1} \le 0\}$. Now $\bar{t}_i \sim t_i \sim \varepsilon_2$ and $\bar{t}_{i+1} < 0$ for $\varepsilon_2 << \varepsilon_1$. If $\bar{t}_i > 0$ and $\bar{t}_{i+1} \le 0$, we the again have the same expression so the difference in (209) is small on this set again, from the continuity of h_0 and q. Otherwise we have $\bar{t}_i \le 0$, and as $t \curvearrowright \bar{t}$, $x \curvearrowright \bar{x}$, $v \curvearrowright \bar{v}$,

$$\bar{t}_{i-1} \curvearrowright t_{i-1} \ge \frac{\varepsilon_1^3}{2C_\xi} > 0,$$

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from (40). We define

$$B_{im}^+ = B_{im} \cap \{0 < t_i \le \varepsilon_2, t_{i+1} \le 0, \text{ but } \bar{t}_i \le 0, \bar{t}_{i-1} > 0\}.$$

By the continuity of h_0 and q, the first term for $h(t,x,v)$ in (209) is close to ($t_i \sim 0$): $\frac{e^{\nu(v)(t_1-t)}}{\tilde{w}(v)} \times$

$$\int_{B_{im}^+} h_0(x_i, v_i)\{\Pi_{j=i+1}^{k-1} d\sigma_j\}\{\tilde{w}(v_i)d\sigma_i\}\Pi_{j=1}^{i-1}\{e^{\nu(v_j)(t_{j+1}-t_j)}d\sigma_j\}$$

$$+ \int_{B_{im}^+} \sum_{j=1}^{i-1} \int_{t_{j+1}}^{t_j} q(\tau, x_j + (\tau - t_j)v_j, v_j)d\Sigma_j(\tau)d\tau. \tag{210}$$

Since $\bar{t}_i \le 0$, and $\bar{t}_{i-1} > 0$, hence $m = i-1$, B_{im}^+ is empty except for $m = i+1$. The second term for $h(\bar{t}, \bar{x}, \bar{v})$ in (209) is given by $\frac{e^{\nu(\bar{v})(\bar{t}_1-\bar{t})}}{\tilde{w}(\bar{v})} \times$

$$\int_{B_{im}^+} h_0(\bar{x}_{i-1} - \bar{t}_{i-1}v_{i-1}, v_{i-1})\{\Pi_{j=i}^{k-1} d\bar{\sigma}_j\}\{\tilde{w}(v_{i-1})e^{-\nu(v_{i-1})\bar{t}_{i-1}}d\bar{\sigma}_{i-1}\}\Pi_{j=1}^{i-2}\{e^{\nu(v_j)(\bar{t}_{j+1}-\bar{t}_j)}d\bar{\sigma}_j\}$$

$$+ \int_{B_{im}^+} \sum_{j=1}^{i-2} \int_{t_{j+1}}^{t_j} q(\tau, \bar{x}_j + (\tau - \bar{t}_j)v_j, v_j)d\bar{\Sigma}_j(\tau)d\tau$$

$$+ \int_0^{\bar{t}_{i-1}} \int_{B_{im}^+} q(\tau, \bar{x}_{i-1} + (\tau - \bar{t}_{i-1})v_{i-1}, v_{i-1})d\bar{\Sigma}_{i-1}(\tau)d\tau.$$

Since $\bar{x}_{i-1} - \bar{t}_{i-1}v_{i-1} \sim \bar{x}_{i-1} - (\bar{t}_{i-1} - \bar{t}_i)v_{i-1} = \bar{x}_i \sim x_i$, $t_i \sim 0$, $\int_0^{\bar{t}_{i-1}} \sim \int_{t_i}^{t_{i-1}}$, the above is simplified as

$$\sim \int_{B_{im}^+} h_0(x_i, v_{i-1})\{\Pi_{j=i}^{k-1} d\sigma_j\}\{\tilde{w}(v_{i-1})e^{\nu(v_{i-1})\{t_i-t_{i-1}\}}d\sigma_{i-1}\}\Pi_{j=1}^{i-2}\{e^{\nu(v_j)(t_{j+1}-t_j)}d\sigma_j\}$$

$$+ \int_{B_{im}^+} \sum_{j=1}^{i-1} \int_{t_{j+1}}^{t_j} q(\tau, x_j + (\tau - t_j)v_j, v_j)d\Sigma_j(\tau)d\tau \tag{211}$$

By the boundary condition (184),

$$h_0(x_i, v_{i-1})\tilde{w}(v_{i-1}) = \int_{\mathcal{V}_i} h_0(x_i, v_i)\tilde{w}(v_i)d\sigma_i \sim \int_{\mathcal{V}_i^{\varepsilon_1} \cap \bar{\mathcal{V}}_i^{\varepsilon_1}} h_0(x_i, v_i)\tilde{w}(v_i)d\sigma_i.$$

For $v_i \in \mathcal{V}_i^{\varepsilon_1} \cap \bar{\mathcal{V}}_i^{\varepsilon_1}$, we have $t_i - t_{i+1} \ge \frac{\varepsilon_1^3}{C_\xi}$. But $t_i \le \varepsilon_2 << \varepsilon_1$ in B_{im}^+, so that $t_{i+1} \le 0$. Hence

$$\mathcal{V}_i^{\varepsilon_1} \cap \bar{\mathcal{V}}_i^{\varepsilon_1} = \{\mathcal{V}_i^{\varepsilon_1} \cap \bar{\mathcal{V}}_i^{\varepsilon_1}\} \cap \{0 < t_i \le \varepsilon_2, t_{i+1} \le 0, \text{ and } \bar{t}_i \le 0, \bar{t}_{i-1} > 0\},$$

i.e., no restriction on $v_i \in \mathcal{V}_i^{\varepsilon_1} \cap \bar{\mathcal{V}}_i^{\varepsilon_1}$ in B_{im}^+, because $t_{i+1} \le 0$ and t_i, $\bar{t}_i$ only depends on $v_1, ... v_{i-1}$, not on v_i from (188). Moreover, since $\int_{\mathcal{V}_i^{\varepsilon_1} \cap \bar{\mathcal{V}}_i^{\varepsilon_1}} d\sigma_i \sim 1$, the first term in (211)

$$\sim \frac{e^{\nu(v)(t_1-t)}}{\tilde{w}(v)} \int_{B_{im}^+} h_0(x_i, v_i)\tilde{w}(v_i)\{\Pi_{j=i+1}^{k-1} d\sigma_j\}\Pi_{j=1}^{i-1}\{e^{\nu(v_j)(t_{j+1}-t_j)}d\sigma_j\},$$

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which is exactly the first term in (210).

CASE 2c: $B_{im} \cap \{t_i > \varepsilon_2, -\varepsilon_2 \le t_{i+1} \le 0\}$. Clearly $\bar{t}_i > 0$ by continuity over B_{im}. If $\bar{t}_{i+1} \le 0$, then we again have $i = m$ in (209) and the difference is small on this set from the continuity of h_0 and q. We therefore only need to consider the set

$$B_{im}^- \equiv B_{im} \cap \{t_i > \varepsilon_2, -\varepsilon_2 \le t_{i+1} \le 0, \text{ but } 0 < \bar{t}_{i+1}, \bar{t}_{i+2} \le 0\}.$$

Since B_{im}^- is empty except for $m = i + 1$, the contribution for $h(\bar{t}, \bar{x}, \bar{v})$ is given by $\frac{e^{\nu(\bar{v})(\bar{t}_1 - \bar{t})}}{\tilde{w}(\bar{v})} \times$

$$\int_{B_{im}^-} h_0(\bar{x}_{i+1} - \bar{t}_{i+1}v_{i+1}, v_{i+1})\{\Pi_{j=i+2}^{k-1}d\sigma_j\}\{\tilde{w}(v_{i+1})e^{-\nu(v_{i+1})\bar{t}_{i+1}}d\sigma_{i+1}\}\Pi_{j=1}^i\{e^{\nu(v_j)(\bar{t}_{j+1}-\bar{t}_j)}d\sigma_j\}$$

$$+ \int_{B_{im}^-} \sum_{j=1}^i \int_{\bar{t}_{j+1}}^{\bar{t}_j} q(\tau, \bar{x}_j + (\tau - \bar{t}_j)v_j, v_j)d\bar{\Sigma}_j(\tau)d\tau$$

$$+ \int_0^{\bar{t}_{i+1}} \int_{B_{im}^-} q(\tau, \bar{x}_{i+1} + (\tau - \bar{t}_{i+1})v_{i+1}, v_{i+1})d\Sigma_{i+1}(\tau)d\tau.$$

Since on B_{im}^-, we have $\bar{t}_{i+1} \to t_{i+1} \sim \varepsilon_2$, the last term is small from the continuity of q and the second term tends to the second term for $h(t, x, v)$ in (209). Since $\bar{t}_{i+1} \sim \varepsilon_2$, $\bar{x}_{j+1} \sim x_{j+1}$ and $\bar{t}_{j+1} \sim \bar{t}_{j+1}$, the first term above takes the form

$$\sim \frac{e^{\nu(v)(t_1-t)}}{\tilde{w}(v)} \int_{B_{im}^-} h_0(x_{i+1}, v_{i+1})\{\Pi_{j=i+2}^{k-1}d\sigma_j\}\{\tilde{w}(v_{i+1})d\sigma_i\}\Pi_{j=1}^i\{e^{\nu(v_j)(t_{j+1}-t_j)}d\sigma_j\}.$$

(212)

On the other hand, consider the first term for $h(t, x, v)$ in (209). Since $x_{i+1} - x_i = -v_i(t_i - t_{i+1})$, and $t_{i+1} \sim \varepsilon_2$, $x_i - t_iv_i - x_{i+1} \sim \varepsilon_2$, by the continuity of h_0 (away from γ_0), it takes the form

$$\sim \frac{e^{\nu(v)(t_1-t)}}{\tilde{w}(v)} \int_{B_{im}} h_0(x_{i+1}, v_i)\{\Pi_{j=i+1}^{k-1}d\sigma_j\}\{\tilde{w}(v_i)e^{-\nu(v_i)t_i}d\sigma_i\}\Pi_{j=1}^{i-1}\{e^{\nu(v_j)(t_{j+1}-t_j)}d\sigma_j\}$$

(213)

Since $\bar{t}_{i+1} \sim \varepsilon_2$ from continuity, for $\varepsilon_2 << \varepsilon_1$, $\bar{t}_{i+2} < 0$, and

$$\mathcal{V}_{i+1}^{\varepsilon_1} \cap \bar{\mathcal{V}}_{i+1}^{\varepsilon_1} \cap \{t_i > \varepsilon_2, -\varepsilon_2 \le t_{i+1} \le 0, \text{ but } 0 < \bar{t}_{i+1}, \bar{t}_{i+2} \le 0\} = \mathcal{V}_{i+1}^{\varepsilon_1} \cap \bar{\mathcal{V}}_{i+1}^{\varepsilon_1},$$

where $\bar{t}_{i+1}$ only depends on $v_1, ...v_i$, but not on v_{i+1} by (188). From the diffuse boundary condition (184) we have

$$h_0(x_{i+1}, v_i)\tilde{w}(v_i) \sim \int_{\mathcal{V}_{i+1}^{\varepsilon_1}\cap\bar{\mathcal{V}}_{i+1}^{\varepsilon_1}\cap B_{im}^-} h_0(x_{i+1}, v_{i+1})\tilde{w}(v_{i+1})d\sigma_{i+1}.$$

Since $\int_{\mathcal{V}_{i+1}^{\varepsilon_1}\cap\bar{\mathcal{V}}_{i+1}^{\varepsilon_1}\cap B_{im}^-} d\sigma_{i+1} \sim 1$, (213) reduces to (212) and we conclude our proof. ∎

4.4.3 Decay for Diffuse Reflection $U(t)$

Theorem 41 *Let $h_0 \in L^\infty$ and assume (21). There exits a unique solution to both the (23) and (27) with the diffuse boundary condition (184). Let $U(t)h_0$ be the solution the the weighted linear Boltzmann equation (27) with the diffuse boundary condition, then there exist $\lambda > 0$ and $C > 0$ such that the exponential decay (142) is valid.*

Proof. Once again, with the L^∞ solution $h(t) = U(t)h_0$ to the weighted linear Boltzmann equation (27), from Lemma 39 and Duhamel principle (30), we deduce from Ukai's trace theorem, $h_\gamma \in L^\infty$. So $f = \frac{h}{w}$ satisfies the original linear Boltzmann equation (23) with $f \in L^2$ and $\int_0^t \|f(s)\|_\gamma^2 ds < \infty$. Hence uniqueness follows for f by using the standard L^2 energy estimate. The well-posedness follows from the exact argument in the proof of Theorem 28. By Lemma 29 and the L^2 decay for the diffuse boundary problem, it suffices to establish the finite time estimate (143).

We apply the double-Duhamel's principle (31). By Lemma 39, the first two terms in (31) are easily bounded by $C_K(t+1)e^{-\nu_0 t}\|h_0\|_\infty$.

We concentrate on the third term

$$\int_0^t \int_0^{s_1} G(t - s_1)K_w G(s_1 - s)K_w h(s)dsds_1. \tag{214}$$

We now fix any point (t, x, v) so that $(x, v) \notin \gamma_0$. From (192) with $q \equiv 0$, the integrand above is given by

$$e^{\nu(v)(s_1 - t)}\mathbf{1}_{\{t_1 \leq s_1\}}\{K_w G(s_1 - s)K_w h(s)\}(s_1, x - tv, v)$$

$$+ \frac{e^{\nu(v)(t_1 - t)}}{\tilde{w}(v)}\int \sum_{l=1}^{k-1}\mathbf{1}_{\{t_l > s_1, t_{l+1} \leq s_1\}}\{K_w G(s_1 - s)K_w h(s)\}(s_1, x_l + (s_1 - t_l)v_l, v_l)d\Sigma_l(s_1)$$

$$+ \frac{e^{\nu(v)(t_1 - t)}}{\tilde{w}(v)}\int \mathbf{1}_{\{t_k > s_1\}}\{G(t - s_1)K_w G(s_1 - s)K_w h(s)\}(t_k, x_k, v_{k-1})d\Sigma_{k-1}(t_k). \tag{215}$$

where $d\Sigma_l(s_1) = \{\Pi_{j=l+1}^{k-1}d\sigma_j\}e^{\nu(v_l)(s_1 - t_l)}\tilde{w}(v_l)d\sigma_l\Pi_{j=1}^{l-1}\{e^{\nu(v_j)(t_{j+1} - t_j)}d\sigma_j\}$, and the exponential factor in $d\Sigma_l(s_1)$ is bounded by $e^{\nu_0(s_1 - t_1)}$. By Lemma 39,

$$\|\{G(t_k - s_1)K_w G(s_1 - s)K_w h(s)\}(t_k)\|_\infty \leq C_K e^{-\frac{\nu_0}{2}(t_k - s)}\|h(s)\|_\infty.$$

Letting $k = k_0(\varepsilon, T_0)$ large in Lemma 37, as in (123), the integral of the last term is bounded by

$$\varepsilon C_K e^{-\frac{\nu_0}{8}t}\sup_{0 \leq t \leq T_0}\{e^{\frac{\nu_0}{8}s}\|h(s)\|_\infty\}. \tag{216}$$

To estimate the first and second terms in (215), we first derive the formula for $K_w G(s_1 - s)K_w h(s)$. Recall the back-time cycles of: $X_{\mathbf{cl}}(s) = x_l - (t_l - s)v_l$.

We denote $t'_{l'} = t_{l'}(s_1, X_{\mathbf{cl}}(s_1), v')$, and $x'_{l'} = X_{\mathbf{cl}}(t_{l'}, (X_{\mathbf{cl}}(s_1), v')$, and $v_{l'} \in \mathcal{V}_{l'}$, for $1 \le l' \le k - 1$. By (192) again,

$$\{K_w G(s_1 - s) K_w h(s)\}(s_1, X_{\mathbf{cl}}(s_1), v_l)$$

$$= \int K_w(v_l, v')\{G(s_1 - s) K_w h(s)\}(s_1, X_{\mathbf{cl}}(s_1), v')\, dv'$$

$$= \int K_w(v_l, v') e^{\nu(v')(s - s_1)} \mathbf{1}_{\{t'_1 \le s\}} \{K_w h(s)\}(s, X_{\mathbf{cl}}(s_1) - (s_1 - s)v', v')dv'$$

$$+ \int K_w(v_l, v') \frac{e^{\nu(v')(t'_1 - s_1)}}{\tilde{w}(v')} \sum_{l'=1}^{k-1} \mathbf{1}_{t'_{l'+1} \le s < t'_{l'}} \{\int K_w(v_{l'}, v'') h(s, x'_{l'} - (t'_{l'} - s)v_{l'}, v'')dv''\}d\Sigma_{l'}dv'$$

$$+ \int K_w(v_l, v') \frac{e^{\nu(v')(t'_1 - s_1)}}{\tilde{w}(v')} \int \mathbf{1}_{t'_k > s} \{G(s_1 - s) K_w h(s)\}(t'_k, x'_k, v_k)d\Sigma_{k-1}(t'_k)dv'. \tag{217}$$

Once again, since $\|G(s_1 - s) K_w h(s)(t'_k, x'_k, v_k)\|_\infty \le C_K e^{\frac{\nu_0}{2}\{t'_k - s\}}\|h(s)\|_\infty$, the last term is bounded by (216) due to Lemma 37, when $k \ge k_0(\varepsilon)$ large.

By inserting the main terms in (217) back to (215), we deduce that, up to $\varepsilon C_K e^{-\frac{\nu_0}{8}t} \sup_{0 \le t \le T_0}\{e^{\frac{\nu_0}{8}s}\|h(s)\|_\infty\}$, the third term (214) in the double-duhamel representation (31) is $\int_0^t \int_0^{s_1} *dsds_1$, where $*$ is

$$e^{\nu(v)(s_1 - t)}\mathbf{1}_{t_1 \le s_1} \int K_w(v, v') e^{\nu(v')(s - s_1)}\mathbf{1}_{t'_1 \le s}\{K_w h(s)\}(s, X_{\mathbf{cl}}(s_1) - (s_1 - s)v, v')dv'$$

$$+ e^{\nu(v)(s_1 - t)}\mathbf{1}_{t_1 \le s_1} \int K_w(v, v') \frac{e^{\nu(v')(t - s_1)}}{\tilde{w}(v')} \sum_{l'=1}^{k-1} \mathbf{1}_{t'_{l'+1} \le s < t'_{l'}} \times$$

$$\times \{\int K_w(v_{l'}, v'') h(s, x'_{l'} - (t'_{l'} - s)v_{l'}, v'')dv''\}d\Sigma'_{l'}(s)dv'$$

$$+ \frac{e^{\nu(v)(t_1 - t)}}{\tilde{w}(v)} \int \sum_{l=1}^{k-1} \mathbf{1}_{t_{l+1} \le s_1 < t_l} K_w(v_l, v') \times \tag{218}$$

$$\times \quad e^{\nu(v')(s - s_1)}\mathbf{1}_{t'_1 \le s} K_w(v', v'') h(s, X_{\mathbf{cl}}(s_1) - (s_1 - s)v', v'')dv''dv'd\Sigma_l(s_1)$$

$$+ \frac{e^{\nu(v)(t_1 - t)}}{\tilde{w}(v)} \int \sum_{l=1, l'=1}^{k-1} \mathbf{1}_{\{t_l > s_1, t_{l+1} \le s_1\}} K_w(v_l, v')d\Sigma_l \times$$

$$\times \mathbf{1}_{\{t'_{l'} > s, t'_{l'+1} \le s\}} \frac{e^{\nu(v')(t'_1 - s_1)}}{\tilde{w}(v')} K_w(v_{l'}, v'') h(s, x'_{l'} - (t'_{l'} - s)v_{l'}, v'')dv''d\Sigma'_{l'}ds_1ds.$$

We now estimate them term by term. For the first term in (218), the back-time trajectories never touch the boundary. This term can be easily estimated as the proof of Theorem 20 for the in-flow case (e.g. (126)) by

$$\varepsilon C_K e^{-\frac{\nu_0}{8}t} \sup_{0 \le t \le T_0}\{e^{\frac{\nu_0}{8}s}\|h(s)\|_\infty\} + C_{\varepsilon, T_0} \int_0^{T_0} \|f(s)\|ds.$$

For the second term in (218), we fix l' and consider $v_{l'}, v'$ and v'' integration. We recall $d\Sigma'_{l'}(s) = \{\Pi_{j=l'+1}^{k-1}d\sigma_j\}e^{\nu(v_{l'})(s - t'_{l'})}\tilde{w}(v_{l'})d\sigma'_{l'}\Pi_{j=1}^{l'-1}\{e^{\nu(v_j)(t'_{j+1} - t'_j)}d\sigma'_j\}$,

and $\tilde{w}(v_{l'})d\sigma_{l'} = \tilde{w}(v_{l'})\mu(v_{l'})\{n(x'_{l'}) \cdot v_{l'}\}dv_{l'}$. The exponential factor in $d\Sigma'_{l'}(s)$ is bounded by $e^{\nu_0(t'_1-s)}$. Notice that with $\tilde{w}(v_{l'})\mu(v_{l'})$ bounded and integrable for $\theta < \frac{1}{4}$. Hence from (45),

$$\int_{|v_{l'}|\geq N} \int |K_w(v_{l'},v'')|dv'' \tilde{w}(v_{l'})\mu(v_{l'})\{n(x'_{l'}) \cdot v_{l'}\}dv_{l'} \qquad (219)$$

$$\leq \quad \frac{C_K}{N} \int_{|v_{l'}|\geq N} \tilde{w}(v_{l'})\mu(v_{l'})\{n(x'_{l'}) \cdot v_{l'}\}dv_{l'} = \frac{C_K}{N}.$$

By similar arguments as in Case 1 (119), Case 2 (122) in the proof of Theorem 20, we only need to consider the case $|v| \leq N$, $|v'| \leq 2N$, and $|v_{l'}| \leq N$ and $|v''| \leq 2N$, for some large N. As in Case 4 in the proof of Theorem 20, we can also use the same approximation (124). Hence, for each fixed l', we only need to control:

$$\int_{t_1}^{t} \int_{\Pi_{j=1}^{l'-1}\mathcal{V}'_j} \left\{ \int_{|v''|\leq 2N,|v_{l'}|\leq N} \int_{t_{l'+1}}^{t'_{l'}} e^{\nu_0(s-t)}|h(s,x'_{l'}-(t'_{l'}-s)v_{l'},v'')|dv''dv'dv_{l'} \right\} \Pi_{j=1}^{l'-1}d\sigma'_j$$

$$\leq \quad \int_{t_{l'+1}}^{t'_{l'}} \mathbf{1}_{\{t'_{l'}-s\geq\frac{1}{N}\}} + \mathbf{1}_{\{t'_{l'}-s\leq\frac{1}{N}\}}$$

$$\leq \quad \int_{t_{l'+1}}^{t'_{l'}} \mathbf{1}_{\{t'_{l'}-s\geq\frac{1}{N}\}} + \frac{C}{N} \sup_{0\leq s\leq T_0} \{e^{-\nu_0(s-t'_{l'})}\|h(s)\|_\infty\}.$$

Here we have used the fact $\int\{\Pi_{j=l'+1}^{k-1}d\sigma_j\} = 1$. Notice that $x'_{l'}$ and $t'_{l'}$ depend only on $t,x,v,v_1,...v_{l'-1}$, not on $v_{l'}$. By making a change of variable $y = x'_{l'} - (t'_{l'}-s)v_{l'}$, for the first part, we use Fubini's theorem and the fact $\int\{\Pi_{j=1}^{l'-1}d\sigma_j\} = 1$ to majorize it as $(f = \frac{h}{w})$:

$$C_{N,T_0} \int_0^{T_0} \left\{ \int_{y\in\Omega,|v''|\leq 2N} |h(s,y,v'')|^2 dy dv'' \right\}^{1/2} = C_{N,T_0,\Omega} \int_0^{T_0} \|f(s)\|ds.$$
$$(220)$$

For the third term in (218), for fixed l, we consider the v_l, v', v'' integration. Recall $d\Sigma_l(s_1) = \{\Pi_{j=l+1}^{k-1}d\sigma_j\}e^{\nu(v_l)(s_1-t_l)}\tilde{w}(v_l)d\sigma_l \Pi_{j=1}^{l-1}\{e^{\nu(v_j)(t_{j+1}-t_j)}d\sigma_j\}$ and the exponential factor is bounded by $e^{\nu_0(t_1-s_1)}$. Because $\tilde{w}(v_l)\mu(v_l)$ is bounded and integrable, as in (219), we can use the arguments in Case 1 (119), Case 2 (122) in the proof of Theorem 20 to reduce to $|v_l| \leq N$, $|v'| \leq 2N$ and $|v''| \leq 3N$. As in the second term, it then suffices to estimate

$$e^{-\nu_0(t-s)} \int_{\Pi_{j=1}^{l-1}\mathcal{V}'_j} \int_{t_{l+1}}^{t_l} \int_s^{s_1} \int_{|v''|\leq 2N,|v'|\leq N} |h(s,X_{\mathbf{cl}}(s_1)-(s_1-s)v',v'')|ds ds_1 dv'' dv' \Pi_{j=1}^{l-1}d\sigma_j$$

which is bounded by (220) by the change of variable $y = X_{\mathbf{cl}}(s_1) - (s_1 - s)v'$.

In the last integral in (218), both back-time diffuse cycles first hit the boundary. For fixed l and l', we consider the $v_l, v_{l'}$ and v', v'' integration. We again

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recall $d\Sigma_l$ and $d\Sigma'_{l'}$ and their exponential factors are bounded by $e^{\nu_0(t_1-s_1)}$ and $e^{\nu_0(t'_1-s)}$ respectively. Notice that both $\tilde{w}(v_l)\mu(v_l)$ and $\tilde{w}(v_{l'})\mu(v_{l'})$ are bounded and integrable. We use twice (219), and as in the reduction to (220), we can reduce to the case of $|v_l| \leq N, |v'| \leq 2N$, and $|v_{l'}| \leq N$, $|v''| \leq 2N$. Therefore, by using the approximation (124) and $\int \{\Pi_{j=l'+1}^{k-1} d\sigma'_j\} = 1$, $\int \{\Pi_{j=l+1}^{k-1} d\sigma_j\} = 1$, we only need to control:

$$\int_{\Pi_{j=1}^{l-1}\mathcal{V}_j} \int_{\Pi_{j=1}^{l'-1}\mathcal{V}'_j} \int_{t_{l+1}}^{t_l} \int_{t'_{l'+1}}^{t'_{l'}} \int_{|v_{l'}|,|v''|\leq 2N} e^{\nu_0(s-t)}|h(s, x'_{l'}-(t'_{l'}-s)v_{l'}, v'')|dv''dv_{l'}dsds_1|\Pi_{j=1}^{l'-1} d\sigma'_j \Pi_{j=1}^{l-1} d\sigma_j,$$

which is again bounded by (220) by the change of variable $y = x'_{l'} - (t'_{l'} - s)v_{l'}$.

Summing over $l, l' \leq k - 1$, and letting N large, we summarize:

$$||h(t)||_\infty \leq C_K\{1+t\}e^{-\frac{\nu_0}{2}t}||h_0||_\infty+\varepsilon C_K e^{-\frac{\nu_0}{8}t} \sup_{0\leq t\leq T_0} \{e^{\frac{\nu_0}{8}s}||h(s)||_\infty\}+C_{N,T_0,\varepsilon} \int_0^{T_0} ||f(s)||ds.$$

Choosing ε small such that $\varepsilon C_K = \frac{1}{2}$, and T_0 large so that $2C_K\{1+T_0\}e^{-\frac{\nu_0}{2}T_0} = e^{-\lambda T_0}$, we deduce (143), and by Lemma 29, we deduce our theorem. $\blacksquare$

5   Nonlinear Exponential Decay

Proof. (of Theorem 1): **Step 1. Existence and Continuity**: Let $h^0 \equiv 0$, we use iteration

$$\{\partial_t + v \cdot \nabla_x + \nu - K_w\}h^{m+1} = w\Gamma(\frac{h^m}{w}, \frac{h^m}{w}) \qquad (221)$$

with $h^{m+1}|_{t=0} = h_0$, $h^{m+1}|_{\gamma^-} = wg$. We further split $h^{m+1} = h_g^{m+1} + h_\Gamma^{m+1}$ for $m \geq 1$, where h_g^{m+1} satisfies the homogeneous linear weighted Boltzmann equation (27)

$$\{\partial_t + v \cdot \nabla_x + \nu - K_w\}h_g^{m+1} = 0, \quad h_g^{m+1}|_{\gamma^-} = wg, \quad h_g^{m+1}|_{t=0} = h_0;$$

while h_Γ^{m+1} satisfies the inhomogeneous linear weighted Boltzmann equation (27) with zero-boundary condition:

$$\{\partial_t + v \cdot \nabla_x + \nu - K_w\}h_\Gamma^{m+1} = w\Gamma(\frac{h^m}{w}, \frac{h^m}{w}), \quad h_\Gamma^{m+1}|_{\gamma^-} = h_\Gamma^{m+1}|_{t=0} = 0.$$

Clearly, from Theorem 20, for some $0 < \lambda < \lambda_0$,

$$\sup_{0\leq t\leq\infty} e^{\lambda t}||h_g^{m+1}(t)||_\infty \leq C\{||h_0||_\infty + \sup_{0\leq t\leq\infty} e^{\lambda_0 t}||wg(t)||_\infty\}.$$

On the other hand, denote $U_0(t,s)$ and $G_0(t,s)$ to be solution operators for linear weighted Boltzmann equation (27) and (29) with *zero* boundary condition

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respectively, then $U_0(t,s) = U_0(t-s, 0)$ and $G_0(t,s) = G(t-s, 0)$ are semigroups. Hence, by the Duhamel Principle,

$$h_\Gamma^{m+1} = \int_0^t U_0(t-s)w\Gamma(\frac{h^m}{w}, \frac{h^m}{w})(s)ds. \qquad (222)$$

To avoid the extra weight function $\nu(v) \sim \{1+|v|\}^\gamma$ in Lemma 9, we further use the Duhamal Principle $U_0(t-s) = G_0(t-s) + \int_s^t G_0(t-s_1)K_w U_0(s_1-s)ds_1$ to bound (222) by

$$||\int_0^t G_0(t-s)w\Gamma(\frac{h^m}{w}, \frac{h^m}{w})(s)ds||_\infty + ||\int_0^t \int_s^t G_0(t-s_1)K_w U_0(s_1-s)w\Gamma(\frac{h^m}{w}, \frac{h^m}{w})(s)ds_1 ds||_\infty. \qquad (223)$$

For any (t,x,v), by Lemma 9:

$$|\{w\Gamma(\frac{h^m}{w}, \frac{h^m}{w})\}(s, x-(t-s)v, v)| \le C\nu(v)||h^m||_\infty^2,$$

By the explicit formula (109) with $g = 0$, we therefore can bound the first term in (223) as

$$|\int_0^t e^{-\nu(v)(t-s)}\mathbf{1}_{t-t_\mathbf{b}(x,v) \le s}\{w\Gamma(\frac{h^m}{w}, \frac{h^m}{w})\}(s, x-(t-s)v, v)ds|$$

$$\le C\int_0^t e^{-\nu(v)(t-s)}\nu(v)||h^m(s)||_\infty^2 ds$$

$$\le C\int e^{-\frac{\nu(v)}{2}(t-s)}\nu(v)ds \times \sup_{0 \le s \le t}\{e^{-\frac{\nu_0}{2}(t-s)}||h^m(s)||_\infty^2\}$$

$$\le Ce^{-\frac{\nu_0}{2}t} \times \sup_s\{e^{\frac{\nu_0}{2}s}||h^m(s)||_\infty\}^2. \qquad (224)$$

Here we have used the fact $\nu(v) \ge \nu_0$, and $\int e^{-\frac{\nu(v)}{2}(t-s)}\nu(v)ds < \infty$.

On the other hand, for the second term in (223), let the semigroup $\tilde{U}(t)h_0$ solve

$$\{\partial_t + v \cdot \nabla_x + \nu - K_{w/\sqrt{1+\rho|v|^2}}\}\{\tilde{U}(t)h_0\} = 0, \qquad (225)$$

with $\{\tilde{U}(t)\tilde{h}_0\}|_{\gamma-} = 0$ and $\tilde{U}(0)\tilde{h}_0 = \tilde{h}_0$. By a direct computation, $\sqrt{1+\rho|v|^2}\tilde{U}(t)$ solves the original linear Boltzmann equation (27). By uniqueness in the L^∞ class, we deduce

$$U(t)h_0 \equiv \sqrt{1+\rho|v|^2}\tilde{U}(t)\{\frac{h_0}{\sqrt{1+\rho|v|^2}}\}. \qquad (226)$$

Therefore, we can rewrite $K_w U(s_1, s)w\Gamma(\frac{h^m}{w}, \frac{h^m}{w})(s)$ to get

$$\int_0^t \int_s^t e^{-\nu_0(t-s_1)}||\{\int K_w(v,v')\{\sqrt{1+\rho|v'|^2}\}\}\tilde{U}(s_1-s)\{\frac{w}{\sqrt{1+\rho|v'|^2}}\Gamma(\frac{h^m}{w}, \frac{h^m}{w})(s)\}||_\infty ds_1 ds dv'.$$

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Since $w^{-2}(1 + |v|)^3 \in L^1$, $\{\frac{w}{\sqrt{1+\rho|v'|^2}}\}^{-2}(1 + |v|) \in L^1$ so that Theorem 20 is valid for $\tilde{U}$. Since $\frac{\nu(v')}{\sqrt{1+\rho|v'|^2}} \le C_\rho$, from the proof of Lemma 9,

$$\int K_w(v,v')\sqrt{1 + \rho|v'|^2}dv' \le C \int K_w(v,v')\{|v - v'| + |v|\}dv' < +\infty. \quad (227)$$

Hence the second term in (223) is bounded

$$C \int_0^t \int_s^t e^{-\nu_0(t-s_1)}\tilde{U}(s_1 - s)||\{\frac{w}{\sqrt{1 + \rho|v'|^2}}\Gamma(\frac{h^m}{w}, \frac{h^m}{w})(s)\}||_\infty ds_1 ds$$

$$\le C \int_0^t \int_s^t e^{-\lambda(t-s_1)}||h^m(s)||_\infty^2 ds_1 ds$$

$$\le Ce^{-\frac{\lambda}{2}t} \times \{\sup_{0 \le s \le \infty} e^{\frac{\lambda}{2}s}||h^m(s)||_\infty\}^2. \quad (228)$$

Collecting terms for both h_g^{m+1} and h_Γ^{m+1}, we obtain for $0 < \lambda < \lambda_0$,

$$\sup_m \sup_{0 \le t \le \infty} \{e^{\lambda t}||h^{m+1}(t)||_\infty\} \le C\{||h_0||_\infty + \sup_{0 \le s \le \infty} e^{\lambda_0 s}||g(s)||_\infty\}.$$

for $||h_0||$ sufficiently small. Moreover, subtracting $h^{m+1} - h^m$ yields:

$$\{\partial_t + v \cdot \nabla_x + \nu - K_w\}\{h^{m+1} - h^m\} = w\{\Gamma(\frac{h^m}{w}, \frac{h^m}{w}) - \Gamma(\frac{h^{m-1}}{w}, \frac{h^{m-1}}{w})\} \quad (229)$$

with zero initial and boundary value. Therefore, splitting

$$\Gamma(\frac{h^m}{w}, \frac{h^m}{w}) - \Gamma(\frac{h^{m-1}}{w}, \frac{h^{m-1}}{w}) = \Gamma(\frac{h^m - h^{m-1}}{w}, \frac{h^m}{w}) - \Gamma(\frac{h^{m-1}}{w}, \frac{h^{m-1} - h^m}{w})$$

we can bound $||h^{m+1}(t) - h^m(t)||_\infty$ as in (224) and (228):

$$C||\int_0^t U_0(t - s)w\Gamma(\frac{h^m - h^{m-1}}{w}, \frac{h^m}{w})(s)ds||_\infty \quad (230)$$

$$+C||\int_0^t U_0(t - s)w\Gamma(\frac{h^{m-1}}{w}, \frac{h^{m-1} - h^m}{w})(s)ds||_\infty$$

$$\le C\sup_s\{e^{\lambda s}\{||h^m(s)||_\infty + ||h^{m-1}(s)||\}\} \times e^{-\lambda t} \times \sup_s\{e^{\lambda s}\{||h^m(s) - h^{m-1}(s)||_\infty\}.$$

Hence h^m is a Cauchy sequence and the limit h is a desired unique solution.

Moreover, if Ω is strictly convex, by Lemma 19, inductively, h^m is continuous in $[0, \infty) \times \{\bar{\Omega} \times \mathbf{R}^3 \setminus \gamma_0\}$. It is straightforward and routine to verify that $w\Gamma(\frac{h^m}{w}, \frac{h^m}{w})$ is continuous in the interior of $[0, \infty) \times \Omega \times \mathbf{R}^3$. Moreover, from Lemma 9, $\sup |\frac{w\Gamma(\frac{h^m}{w}, \frac{h^m}{w})}{\nu}|$ is also finite. We therefore deduce that h^{m+1} and hence h is also continuous in $[0, \infty) \times \{\bar{\Omega} \times \mathbf{R}^3 \setminus \gamma_0\}$ from the uniform convergence.

Step 2. Uniqueness and Positivity. Assume that there is another solution $\tilde{h}$ to the full Boltzmann equation with the same initial and boundary condition as h. Assume that $\sup_{0 \le t \le T_0} ||\tilde{h}(t)||_\infty$ is sufficiently small. Then

$$\{\partial_t + v \cdot \nabla_x + \nu - K_w\}\{h - \tilde{h}\} = w\{\Gamma(\frac{h}{w}, \frac{h}{w}) - \Gamma(\frac{\tilde{h}}{w}, \frac{\tilde{h}}{w})\},$$

so that $||h(t) - \tilde{h}(t)||_\infty$ is bounded by as in step 1:

$$C||\int_0^t U_0(t-s)w\Gamma(\frac{h-\tilde{h}}{w}, \frac{h}{w})(s)ds||_\infty + C||\int_0^t U_0(t-s)w\Gamma(\frac{\tilde{h}}{w}, \frac{h-\tilde{h}}{w})(s)ds||_\infty.$$
$$\le \quad C_{T_0} \sup_{0 \le s \le T_0} \{\{||h(s)||_\infty + ||\tilde{h}(s)||\}\} \sup_{0 \le s \le T_0} \{||h(s) - \tilde{h}(s)||_\infty.$$

This implies that $\sup_{0 \le t \le T_0} ||h(t) - \tilde{h}(t)||_\infty \equiv 0$ if $\sup_{0 \le t \le T_0} ||\tilde{h}(t)||_\infty$ and $\sup_{0 \le t \le T_0} ||h(t)||_\infty$ sufficiently small.

Finally, we address the positivity of the $F = \mu + \sqrt{\mu}f$. We use a different approximation for the original Boltzmann equation (1)

$$\{\partial_t + v \cdot \nabla_x\}F^{m+1} + \nu(F^m)F^{m+1} = Q_{\text{gain}}(F^m, F^m), \qquad (231)$$

with the inflow boundary condition $F^{m+1}|_{\gamma^-} = \mu + \sqrt{\mu}g \ge 0$ and $F^{m+1}|_{t=0} = \mu + \sqrt{\mu}f_0$, starting with $F^0 \equiv \mu$. Here

$$\nu(F^m) = \int F^m(v)|v - u|^\gamma du d\omega.$$

In terms of $f^m = \frac{F^m - \mu}{\sqrt{\mu}}$, by (9) and (10), we have

$$\{\partial_t + v \cdot \nabla_x + \nu\}f^{m+1} = Kf^m + \Gamma_{\text{gain}}(f^m, f^m) - \Gamma_{\text{loss}}(f^m, f^{m+1}). \qquad (232)$$

It is routine and standard to show that $h^m = wf^m$ is convergent in L^∞, locally in time $[0, T_0]$, where T_0 depends on $||h_0||_\infty = ||\frac{F_0 - \mu}{\sqrt{\mu}}||_\infty$ and $\sup_{0 \le t \le T_0} ||g(t)||_\infty$. By induction on m, we can show that if $F^m \ge 0$, the $Q_{\text{gain}}(F^m, F^m) \ge 0$. Denote the integration factor as

$$I(t, s) = e^{-\int_s^t \nu(F^m)(\tau, X(\tau), V(\tau))d\tau}, \qquad (233)$$

so that $\frac{d}{ds}\{I(t, s)F^{m+1}\} = I(t, s)Q_{\text{gain}}(F^m, F^m)$ almost everywhere along the back-time trajectory $[X(s), V(s)]$ of t, x, v, inside $\bar{\Omega}$. As in (109), we express $F^{m+1}(t, x, v) =$

$$\mathbf{1}_{t-t_{\mathbf{b}} \le 0}\{I(t, 0)F_0(x - vt, v) + \int_0^t I(t, s)Q_{\text{gain}}(F^m, F^m)(X(s), V(s))ds\} + \mathbf{1}_{t-t_{\mathbf{b}} > 0} \times$$

$$\{I(t, t - t_{\mathbf{b}})\{\mu + \sqrt{\mu}g\}(t - t_{\mathbf{b}}, x - vt_{\mathbf{b}}, v) + \int_{t-t_{\mathbf{b}}}^t I(t, s)Q_{\text{gain}}(F^m, F^m)(X(s), V(s)))ds\}$$

so that $F^{m+1} \geq 0$ on $[0, T_0]$. This implies that $F \geq 0$ in the limit with $h = \frac{F - \mu}{\sqrt{\mu}} \in L^\infty$. By the uniqueness of our solutions with this class, F is the same solution we constructed earlier. We obtain $F \geq 0$ for all time by repeating $[0, T_0], [T_0, 2T_0]...[kT_0, (k+1)T_0]$, from the uniform bound of $\sup_t ||h(t)||_\infty$. ∎

Proof. (of Theorem 2 and Theorem 3): We use the same iteration (221) with either bounce-back or specular reflection for h^{m+1}. By the Duhamel Principle,

$$h^{m+1} = U(t)h_0 + \int_0^t U(t-s)w\Gamma(\frac{h^m}{w}, \frac{h^m}{w})(s)ds. \qquad (234)$$

Applying either Theorems 28 or 35 respectively, by Lemma 9, we have

$$||h^{m+1}(t)||_\infty \leq Ce^{-\lambda t}||h_0||_\infty + ||\int_0^t U(t-s)w\Gamma(\frac{h^m}{w}, \frac{h^m}{w})(s)ds||_\infty.$$

To avoid the extra weight function $\nu(v) \sim \{1 + |v|\}^\gamma$ in Lemma 9, we use $U(t-s) = G(t-s) + \int_s^t G(t-s_1)K_w U(s_1 - s)ds_1$ to estimate the second term above. For any $(t, x, v) \notin \gamma_0$, and its back-time cycle $(X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))$, we use (131) and (159) respectively to get

$$|\int_0^t G(t-s)\{w\Gamma(\frac{h^m}{w}, \frac{h^m}{w})(s)\}ds| = |\int_0^t e^{-\nu(v)(t-s)}\{w\Gamma(\frac{h^m}{w}, \frac{h^m}{w})(s, X(s), V(s))\}ds|$$

$$\leq \quad C|\int_0^t e^{-\nu(v)(t-s)}\nu(v)||h^m(s)||_\infty^2 ds \leq Ce^{-\frac{\nu_0}{2}t} \times \{\sup_{0 \leq s \leq \infty} e^{\frac{\nu_0}{2}s}||h^m(s)||_\infty\}^2$$

where we have used Lemma 9 and (224).

On the other hand, for the second term, we use $\tilde{U}$ as in (225) with either the bounce-back or specular reflection. Hence (226) still holds. Since $w^{-2}(1+|v|)^3 \in L^1$, Theorems 28 or 35 are valid for $\tilde{U}$, and we get from (228)

$$\int_0^t \int_s^t e^{-\nu_0(t-s_1)}||K_w U(s_1 - s)w\Gamma(\frac{h^m}{w}, \frac{h^m}{w})(s)||_\infty ds ds_1$$

$$\leq \quad C\int_0^t \int_s^t e^{-\nu_0(t-s_1)}\left\{\int K_w(v, v')\sqrt{1 + \rho|v'|^2}dv'\right\}\tilde{U}(s_1 - s)||\{\frac{w\Gamma(\frac{h^m}{w}, \frac{h^m}{w})(s)\}}{\sqrt{1 + \rho|v'|^2}}||_\infty ds ds_1$$

$$\leq \quad C\int_0^t \int_s^t e^{-\nu_0(t-s_1)}\tilde{U}(s_1 - s)||\frac{w\Gamma(\frac{h^m}{w}, \frac{h^m}{w})(s)}{\sqrt{1 + \rho|v'|^2}}||_\infty ds ds_1$$

$$\leq \quad Ce^{-\frac{\lambda}{2}t} \times \{\sup_{0 \leq s \leq \infty} e^{\frac{\lambda}{2}s}||h^m(s)||_\infty\}^2, \qquad (235)$$

by (227). This implies that $\sup_m \sup_{0 \leq t \leq \infty}\{e^{\frac{\lambda}{2}t}||h^{m+1}(t)||_\infty\} \leq C||h_0||_\infty$ for $||h_0||$ sufficiently small. Moreover, by subtracting $h^{m+1} - h^m$, by (229) and (230), we deduce that h^m is a Cauchy sequence and the limit h is the desired unique solution. If Ω is strictly convex, inductively, by Lemma 25 and 32 respectively, h^m is continuous in $[0, \infty) \times \{\bar{\Omega} \times \mathbf{R}^3 \setminus \gamma_0\}$, and $w\Gamma(\frac{h^m}{w}, \frac{h^m}{w})$ is continuous in $[0, \infty) \times \Omega \times \mathbf{R}^3$. Moreover, from Lemma 9, $\sup|\frac{w\Gamma(\frac{h^m}{w}, \frac{h^m}{w})}{\nu}|$ is also finite. We

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therefore deduce that h is also continuous in $[0,\infty) \times \{\bar{\Omega} \times \mathbf{R}^3 \setminus \gamma_0\}$ from the uniform convergence.

As for the positivity of F, we follow the argument in the proof of the inflow case by using a different iterative scheme (231) with either bounce-back or specular reflection boundary conditions, so that $f^m = \frac{F^m - \mu}{\sqrt{\mu}}$ satisfies (232). Again, it follows from a routine procedure to show that $h^m = w f^m$ is a Cauchy sequence, local in time $[0, T_0]$. Assume $F^m \geq 0$. For any $(t, x, v) \notin \gamma_0$, we denote its back-time cycle (either bounceback or specular) as $[X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s)]$, and $t_1, t_2, ... t_l > 0$, so that $t_{l+1} \leq 0$. Recall (233), so that $\frac{d}{ds}\{I(t, s)F^{m+1}\} = I(t, s)Q_{\text{gain}}(F^m, F^m)$ almost everywhere along the back-time cycles $[X(s), V(s)]$ of t, x, v, inside $\bar{\Omega}$. As in Lemmas 131 and 159, we can express $F^{m+1}(t, x, v)$ as

$$I(t, t_1)F^{m+1}(t_1, x_1, v_1) + \int_{t_1}^{t} I(s, t)Q_{\text{gain}}(F^m, F^m)(X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s)ds \geq I(t, t_1)F^{m+1}(t_1, x_1, v_1).$$

For $1 \leq i \leq l$, similarly,

$$F^{m+1}(t_i, x_i, v_i) = I(t_i, t_{i+1})F^{m+1}(t_{i+1}, x_{i+1}, v_{i+1}) + \int_{t_1}^{t} I(s, t_i)Q_{\text{gain}}(F^m, F^m)(X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s)ds,$$

$$F^{m+1}(t_l, x_l, v_l) = I(t_l, 0)F_0(X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0)) + \int_0^{t_l} I(s, 0)Q_{\text{gain}}(F^m, F^m)(X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s)ds \geq 0.$$

Hence by an induction over i, $F^{m+1} \geq 0$ and we deduce $F \geq 0$ by the uniqueness as in the proof of the in-flow case. $\blacksquare$

Proof. of Theorem 4: We use the same iteration (221) together with the diffusive boundary condition (184). By (234), we further bound its last term by the Duhamal principle as before:

$$||\int_0^t G(t-s)w\Gamma(\frac{h^m}{w}, \frac{h^m}{w})(s)ds||_\infty + ||\int_0^t \int_s^t G(t-s_1)K_w U(s_1-s)w\Gamma(\frac{h^m}{w}, \frac{h^m}{w})(s)ds ds_1||_\infty.$$

We estimate the first term above in two parts:

$$|\int_0^t G(t-s)\{w\Gamma(\frac{h^m}{w}, \frac{h^m}{w})(s)\}ds| \leq |\int_{t-1}^t| + |\int_0^{t-1}|$$

For $\int_{t-1}^t$, we use estimate (195) and Lemma 9 to get

$$|\int_{t-1}^t e^{-\nu_0(t-s)}\mathbf{1}_{\{t_1 \leq s\}}\{\frac{w\Gamma(\frac{h^m}{w}, \frac{h^m}{w})(s)}{\tilde{w}}\}ds| + |\int_{t-1}^t e^{-\nu(v)(t-s)}\mathbf{1}_{\{t_1 \leq s\}}\{w\Gamma(\frac{h^m}{w}, \frac{h^m}{w})(s)\}ds|$$

$$\leq C \sup |\int_{t-1}^t e^{-\nu(v)(t-s)}\nu(v)ds| \times ||h^m||_\infty^2 + Ce^{-\frac{\nu_0}{2}t}\sup\{e^{\frac{\nu_0}{2}s}||h(s)||_\infty\}^2$$

$$\leq Ce^{-\frac{\nu_0}{2}t}\sup\{e^{\frac{\nu_0}{2}s}||h(s)||_\infty\}^2.$$

For $\int_0^{t-1}$, we use (196) to get extra decay in v as:

$$\int_0^{t-1} e^{-\frac{\nu_0}{2}(t-s)}\{||\frac{w\Gamma(\frac{h^m}{w}, \frac{h^m}{w})(s)}{\tilde{w}}||_\infty + ||e^{-\nu(v)}w\Gamma(\frac{h^m}{w}, \frac{h^m}{w})(s)||_\infty\}ds \leq Ce^{-\frac{\nu_0}{2}t}\sup\{e^{\frac{\nu_0}{2}s}||h(s)||_\infty\}^2.$$

We have used the fact $\frac{1}{\tilde{w}}$ decays exponentially by (187).

On the other hand, for the second term, we define $\tilde{U}$ again in (225) with new weight $w_1 = \frac{w}{\{1+\rho|v|^2\}^{\frac{1}{2}}}$ and the new diffuse boundary condition as

$$\{\tilde{U}\tilde{h}_0\}(t,x,v)|_{\gamma_-} = \frac{1}{\tilde{w}_1}\int_{v'\cdot n(x)>0}\{\tilde{U}\tilde{h}_0\}(t,x,v')\tilde{w}_1(v')d\sigma.$$

Hence (226) still holds. By letting ρ further sufficienlty small in (21), we can apply Theorem 41 for the diffusive $\tilde{U}$ exactly as in (235). Hence, $\sup_{m,0\le t\le\infty}\{e^{\lambda t}||h^{m+1}(t)||_\infty\} \le C||h_0||_\infty$ for $||h_0||$ sufficiently small. As before, we can construct the desired unique solution and establish continuity when Ω is convex.

As for the positivity of F, we follow use the different iterative scheme (231) with diffuse boundary condition

$$F^{m+1}(t,x,v)|_{\gamma_-} = c_\mu\mu(v)\int F^{m+1}(t,x,v')\{n\cdot v'\}dv',$$

and $f^m = \frac{F^m-\mu}{\sqrt{\mu}}$ satisfies (232). Again, it follows from a routine procedure that $h^m = wf^m$ is a Cauchy sequence, local in time $[0,T_0]$. Assume $F^m \ge 0$ and recall (233) so that $\frac{d}{ds}\{I(t,s)F^{m+1}\} = I(t,s)Q_{\text{gain}}(F^m,F^m)$ almost everywhere along the back-time trajectory $[X(s),V(s)]$ of t,x,v, inside $\bar{\Omega}$. Recall (186). For any (t,x,v), consider its back-time diffusive cycle $(X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))$. By a similar derivation as in (192), if $t_1(t,x,v)\le 0$,

$$F^{m+1}(t,x,v) = I(t,0)F_0(x-vt,v)+\int_0^t I(s,0)Q_{\text{gain}}(F^m,F^m)(X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))ds \ge 0.$$

If $t_1(t,x,v) > 0$, then for $k \ge 2$,

$$F^{m+1}(t,x,v) = \int_{t_1}^t I(s,t_1)Q_{\text{gain}}(F^m,F^m)(X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))ds+I(t,t_1)c_\mu\mu(v)\int_{\Pi_{j=1}^{k-1}\mathcal{V}_j}H^m$$

where H^m is given by

$$\sum_{l=1}^{k-1}\mathbf{1}_{\{t_l>0,t_{l+1}\le 0\}}F_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))d\Sigma_l^m(0)$$

$$+\sum_{l=1}^{k-1}\int_{t_{l+1}}^{t_l}\mathbf{1}_{\{t_{l+1}>0\}}Q_{\text{gain}}(F^m,F^m)(X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))d\Sigma_l^m(s)ds$$

$$+\sum_{l=1}^{k-1}\int_s^{t_l}\mathbf{1}_{\{t_l>0,t_{l+1}\le 0\}}Q_{\text{gain}}(F^m,F^m)(X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))d\Sigma_l^m(s)ds$$

$$+\mathbf{1}_{\{t_k>0\}}F^{m+1}(t_k,x_k,v_{k-1})d\Sigma_{k-1}^m(t_k),$$

where $d\Sigma_l(s) = \{\Pi_{j=l+1}^{k-1}d\sigma_j\}\{I(s,t_l)[n(x_l)\cdot v_l]dv_l\}\Pi_{j=1}^l\{I(t_j,t_{j+1})d\sigma_j\}$. For any $\varepsilon > 0$, by Lemma 37, $\int_{\Pi_{j=1}^{k-2}\mathcal{V}_j}\mathbf{1}_{\{t_{k-1}>0\}}\Pi_{j=1}^{k-2}d\sigma_j < \varepsilon$ for k large. Notice that

$\{t_k > 0\} \subset \{t_{k-1} > 0\}$, and by (233), $I(s, t_l) \leq C(\sup_{m, 0 \leq s \leq T_0} ||F^m(s)||_\infty, T_0)$.
Since $F_0 \geq 0$ and $Q_{\text{gain}}(F^m, F^m) \geq 0$, we conclude $F^{m+1}(t, x, v) \geq$

$$C(\sup_{m, 0 \leq s \leq T_0} ||F^m(s)||_\infty, T_0) \int_{\Pi_{j=1}^{k-1} \mathcal{V}_j} \mathbf{1}_{\{t_k > 0\}} F^{m+1}(t_k, x_k, v_k) dv_{k-1} \Pi_{j=1}^{k-2} d\sigma_j$$

$$\geq -C(\sup_{m, 0 \leq s \leq T_0} ||h^m(s)||_\infty, T_0) \int_{\Pi_{j=1}^{k-2} \mathcal{V}_j} \mathbf{1}_{\{t_{k-1} > 0\}} \left\{ \int_{\mathcal{V}_{k-1}} \{\mu + \sqrt{\mu} f^{m+1}\}(t_k, x_k, v_k) dv_{k-1} \right\} \Pi_{j=1}^{k-2} d\sigma_j$$

$$\geq -C(\sup_{m, 0 \leq s \leq T_0} ||h^m(s)||_\infty, T_0)\varepsilon.$$

Since ε is arbitrary, we deduce that $F^{m+1} \geq 0$ and this conclude the positivity of F over $[0, T_0]$. We then conclude $F \geq 0$ by the uniqueness. ∎

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6   References

[A] Arkeryd, L, On the strong L^1 trend to equilibrium for the Boltzmann equation'. Studies in Appl. Math. (1992) 87, 283-288.

[AC] Arkeryd, L; Cercignani, C, A global existence theorem for the initial boundary value problem for the Boltzmann equation when the boundaries are not isothermal. Arch. Rational Mech. Anal. (1993) 125, 271-288.

[AEMN] Arkeryd, L., Esposito, R., Marra R. and Nouri, A., Stability of the laminar solution of the Boltzmann equation for the Benard problem. Preprint 2007.

[AEP] Arkeryd, L.; Esposito, R.; Pulvirenti, M.: The Boltzmann equation for weakly inhomogeneous data. Comm. Math. Phys. 111 (1987), no. 3, 393–407.

[ADVW] Alexandre, R., Desvillettes, L., Villani, C. and Wennberg, B., Entropy dissipation and long-range interactions. Arch. Ration. Mech. Anal. 152 (2000), 4, 327-355.

[AH] Arkeryd, L.; Heintz, A.: On the solvability and asymptotics of the Boltzmann equation in irregular domains. Comm. Partial Differential Equations 22 (1997), no. 11-12, 2129–2152.

[BP] Beals, R.; Protopopescu, V: Abstract time-depedendent transport equations. J. Math. Anal. Appl. (1987) 212, 370-405.

[C1] Cercignani, C.: The Boltzmann Equation and Its Application, Springer-Verlag, 1988.

[C2] Cercignani, C., Equilibrium States and the trend to equlibrium in a gas according to the Boltzmann equation. Rend. Mat. Appl. (1990) 10, 77-95.

[C3] Cercignani, C., On the initial-boundary value problem for the Boltzmann equation. Arch. Rational Mech. Anal. (1992) 116, 307-315.

[CC] Cannoe, R.; Cercignani, C, A trace theorem in kinetic theory. Appl. Math. Letters. (1991) 4, 63-67.

[CIP] Cercignani, C., Illner, R. and Pulvirenti, M., The Mathematical Theory of Dilute Gases, Springer-Verlag, 1994.

[D] Deimling, K., Nonlinear Functional Analysis. Springer-Verlag, 1988.

[De] Desvillettes, L: Convergence to equilibrium in large time for Boltzmann and BGK equations. Arch. Rational Mech. Anal. (1990) 110, 73-91.

[DeV] Desvillettes, L.; Villani, C., On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation. Invent. Math. 159 (2005), no. 2, 245–316.

[DL2] Diperna, R., Lions, P-L., On the Cauchy problem for the Boltzmann equation. Ann. Math., (1989) 130, 321-366.

[DL2] Diperna, R., Lions, P-L., Global weak solution of Vlasov-Maxwell systems, Comm. Pure Appl. Math., 42, 729-757 (1989).

[EGM] Esposito, L., Guo, Y., Marra, R., The Vlasov-Boltzamann system for phase transition. In preparation.

[G1] Guo, Y., The Vlasov-Poisson-Boltzmann system near Maxwellians. Comm. Pure Appl. Math. 55 (2002) no. 9, 1104-1135.

[G2] Guo, Y., The Vlasov-Maxwell-Boltzmann system near Maxwellians. Invent. Math. 153 (2003), no. 3, 593–630.

[G3] Guo, Y., Singular solutions of the Vlasov-Maxwell system on a half line. Arch. Rational Mech. Anal. 131 (1995), no. 3, 241–304.

[G4] Guo, Y., Regularity for the Vlasov equations in a half-space. Indiana Univ. Math. J. 43 (1994), no. 1, 255–320.

[GL] Glassey, R., The Cauchy Problems in Kinetic Theory, SIAM, 1996.

[GS] Guo, Y., Strain, R., The relativistic Maxwell-Boltzmann system. In preparation.

[Gr1] Grad, H.: Principles of the kinetic theory of gases. Handbuch der Physik, XII, 205-294 (1958).

[Gr2] Grad, H.: Asymptotic theory of the Boltzmann equation. II. Rarefied gas dynamics, 3rd Symposium, 26-59, Paris, 1962.

[Gui] Guiraud, J. P. : An $H-$theorem for a gas of rigid spheres in a bounded domain. Theories cinetique classique et relativistes, (1975) G. Pichon, ed., 29-58, CNRS, Paris.

[H] Hamdache, K: Initial boundary value problems for Boltzmann equation. Global existence of week solutions. Arch. Rational Mech. Anal. (1992) 119, 309-353.

[HH] Hwang, H-J., Regularity for the Vlasov-Poisson system in a convex domain. SIAM J. Math. Anal. 36 (2004), no. 1, 121–171.

[HV] Hwang, H-J., Velazquez, J., Global existence for the Vlasov-Poisson system in bounded domain. Preprint 2007.

[LY] Liu, T-P.; Yu, S-H., Initial-boundary value problem for one-dimensional wave solutions of the Boltzmann equation. Comm. Pure Appl. Math. 60 (2007), no. 3, 295–356.

[M] Maslova, N. B. Nonlinear Evolution Equations, Kinetic Approach. Singapore, World Scientific Publishing, 1993.

[Mi] Mischler, S., On the initial boundary value problem for the Vlasov-Poisson-Boltzmann system. Commun. Math. Phys. 210 (2000) 447-466.

[MS] Masmoudi, N.; Saint-Raymond, L., From the Boltzmann equation to the Stokes-Fourier system in a bounded domain. Comm. Pure Appl. Math. 56 (2003), no. 9, 1263–1293.

[S] Shizuta, Y., On the classical solutions of the Boltzmann equation, Comm. Pure Appl. Math., 36 (1983) 705-754.

[SA] Shizuta, Y., Asano, K., Global solutions of the Boltzmann equation in a bounded convex domain. Proc. Japan Acad. (1977) 53A, 3-5.

[SG] Strain, R. Guo, Y., Exponential decay for soft potentials near Maxwellians. Arch. Rational Mech. Anal., in press, 2007.

[U1] Ukai, S., Solutions of the Boltzmann equation. Pattern and waves-Qualitative Analysis of Nonlinear Differential Equations, (1986) 37-96.

[U2] Ukai, S., private communications.

[US] Ukai, S., Asano, K., On the initial boundary value problem of the linearized Boltzmann equation in an exterior domain. Proc. Japan Acad. (1980) 56, 12-17.

[V] Villani, C., Hypocoercivity. Memoir of AMS, to appear.

[Vi] Vidav, I., Spectra of perturbed semigroups with applications to transport theory. J. Math. Anal. Appl., 30 (1970) 264-279.

[YZ] Yang, T.; Zhao, H-J., A half-space problem for the Boltzmann equation with specular reflection boundary condition. Comm. Math. Phys. 255 (2005), no. 3, 683–726.